Risk Matters: Breaking Certainty Equivalence

Web Appendix

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C The discrete-time model

This appendix introduces an equivalent discrete-time version of the prototype RBC model studied in the paper. The model follows closely that in Jermann (1998). Table C1 gives a summary of the model setup in continuous and discrete time. We also provide a summary of the perturbation method for discrete-time economies in the spirit of Schmitt-Grohe and Uribe (2004); Fernandez-Villaverde et al. (2016), and stress how certainty equivalence results from a first-order approximation. Finally, we discuss the concept of risky steady state and how to approximate it based on the work by de Groot (2013).

C.1 The social planner's problem

Consider the problem faced by a social planner with preferences over streams of consumption, C_t, which are summarized by the expected present discounted value of a representative agent's life time utility

$$\tilde{U}_0 \equiv \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \right], \tag{C.1}$$

where $\beta \in (0,1)$ is the subjective discount factor. We further assume that consumption is a non-negative choice that cannot fall below a subsistence level, $C_t \geq X_t$, where X_t denotes internal habits in consumption. Following Grishchenko (2010), the household's internal habit is defined as

$$X_t = \tilde{b} \sum_{s=0}^{t-1} (1-\tilde{a})^{t-s-1} C_s$$

or equivalently,

$$X_t = \tilde{b} C_{t-1} + (1-\tilde{a}) X_{t-1}. \tag{C.2}$$

The parameters $\tilde{a}$ and $\tilde{b}$ share the same interpretation as in the main text, although a tilde on top of the parameters indicates that their value might not be the same due to the discrete-time nature of the problem. Note that once again the household preferences collapse to the standard time-separable case if $X_0 = \tilde{b} = 0$.

The aggregate output of the economy is produced using the Cobb-Douglas technology

$$Y_t = \exp(A_t) K_t^{\alpha} L_t^{1-\alpha}, \tag{C.3}$$

where K_t is the aggregate capital stock, and L_t is the perfectly inelastic labor supply (normalized to one $\forall t \geq 0$). The former accumulates according to

$$K_{t+1} = \Phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta) K_t, \quad K_0 > 0, \tag{C.4}$$

where

$$\Phi(I_t/K_t) = \frac{a_1}{1-1/\xi}\left(\frac{I_t}{K_t}\right)^{1-1/\xi} + a_2, \tag{C.5}$$

represents adjustment costs of adjusting capital. On the other hand, total factor productivity (TFP), A_t, is assumed to follow the AR(1) process

$$A_{t+1} = \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1} \quad A_0 > 0, \tag{C.6}$$

where $\tilde{\rho}_A \in (0,1)$ measures the degree of persistence of technology, $\tilde{\sigma}_A > 0$ its volatility, and $\epsilon_{A,t} \sim \mathcal{N}(0,1)$ is a productivity shock. Finally, the economy satisfies the aggregate resource constraint

$$Y_t = C_t + I_t. \tag{C.7}$$

The problem faced by the social planner is that of choosing the time path for consumption that maximizes (C.1) subject to the dynamic constraints (C.2),(C.4), and (C.6), and the static constraints (C.3), (C.5), and (C.7):

$$\widetilde{V}(K_0, A_0, X_0) = \max_{\{C_t \geq X_t \in \mathbb{R}^+\}_{t=0}^{\infty}} \tilde{U}_0 \quad \text{s.t.} \quad (C.2) - (C.7), \tag{C.8}$$

in which $C_t \geq X_t \in \mathbb{R}^+$ denotes the control variable at time $t \in \mathbb{Z}$, and $\widetilde{V}_0 \equiv \widetilde{V}(K_0, X_0, A_0)$ the value of the optimal plan (value function) from the perspective of time $t = 0$. For any $t \in \{0, 1, 2, \dots\}$, a necessary condition for optimality is given by the *Bellman equation*

$$\widetilde{V}(K_t, A_t, X_t) = \max_{C_t \geq X_t \in \mathbb{R}^+} \left\{ \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \widetilde{V}(K_{t+1}, A_{t+1}, X_{t+1}) \right\} \tag{C.9}$$

subject to

$$\begin{aligned} K_{t+1} &= \Phi\left(\frac{\exp(A_t)K_t^{\alpha} - C_t}{K_t}\right) K_t + (1-\delta) K_t \\ X_{t+1} &= \tilde{b} C_t + (1-\tilde{a}) X_t \\ A_{t+1} &= \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1}. \end{aligned}$$

The first order condition for an interior solution is

$$(C_t - X_t)^{-\gamma} + \tilde{b}\beta \mathbb{E}_t\left[\tilde{V}_{X,t+1}\right] = \Phi'\left(\frac{\exp(A_t)K_t^{\alpha} - C_t}{K_t}\right) \beta \mathbb{E}_t\left[\tilde{V}_{K,t+1}\right], \tag{C.10}$$

where $\widetilde{V}_{K,t+1} \equiv \widetilde{V}_K(K_{t+1}, X_{t+1}, A_{t+1})$, $\widetilde{V}_{X,t+1} \equiv \widetilde{V}_X(K_{t+1}, X_{t+1}, A_{t+1})$, and $\widetilde{V}_{A,t+1} \equiv \widetilde{V}_A(K_{t+1}, X_{t+1}, A_{t+1})$ are the partial derivatives of the value function with respect to each of the states. Equation (C.10) makes optimal consumption an implicit function of the state variables, $C_t^{\star} = C(K_t, A_t, X_t)$.

By means of the Envelope theorem, the costate variable with respect to capital is defined by

$$\widetilde{V}_{K,t} = \beta \left(\Phi' \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) \left((\alpha - 1) \exp(A_t) K_t^{\alpha-1} + \frac{C_t}{K_t} \right) + \Phi \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) + 1 - \delta \right) \mathbb{E}_t \left[\widetilde{V}_{K,t+1} \right],$$

while with respect to the habit by

$$\widetilde{V}_{X,t} = -\left(C_t - X_t \right)^{-\gamma} + \left(1 - \tilde{a} \right) \beta \mathbb{E}_t \left[\widetilde{V}_{X,t+1} \right].$$

A solution to the planner's problem is given by the sequence $\left\{ \widetilde{V}_{K,t}, \widetilde{V}_{X,t}, K_t, X_t, A_t \right\}_{t=0}^{\infty}$ that solves the boundary value problem (with appropriate transversality conditions) characterized by the system of equilibrium stochastic difference equations:

$$\widetilde{V}_{K,t} = \beta \left(\Phi' \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) \left((\alpha - 1) \exp(A_t) K_t^{\alpha-1} + \frac{C_t}{K_t} \right) + \Phi \left(\frac{\exp(A_t) K_t^{\alpha} - C_t}{K_t} \right) + 1 - \delta \right) \mathbb{E}_t \left[\widetilde{V}_{K,t+1} \right] \tag{C.11}$$

$$\widetilde{V}_{X,t} = -\left(C_t - X_t \right)^{-\gamma} + \left(1 - \tilde{a} \right) \beta \mathbb{E}_t \left[\widetilde{V}_{X,t+1} \right] \tag{C.12}$$

$$X_{t+1} = \tilde{b} C_t + (1 - \tilde{a}) X_t \tag{C.13}$$

$$K_{t+1} = \Phi \left(\left(\exp(A_t) K_t^{\alpha} - C_t \right) / K_t \right) K_t + (1 - \delta) K_t \tag{C.14}$$

$$A_{t+1} = \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1} \tag{C.15}$$

together with initial conditions $K(0) = K_0$, $X(0) = X_0$, and $A(0) = A_0$, and where C_t solves the non-linear algebraic equation in (C.10).

Table C1 gives a summary of the model setup in continuous and discrete time.

C.2 Deterministic steady state

In the absence of uncertainty ($\tilde{\sigma}_A = 0$), the deterministic steady state is defined as an equilibrium in which all variables in the economy are constant. Hence, given the assumptions on the capital adjustment cost function in (C.5), the deterministic steady

	Continuous-time	Discrete-time
Objective function	$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$	$\mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\right]$
Market clearing	$\exp(A_t)K_t^\alpha L^{1-\alpha} = C_t + I_t$	$\exp(A_t)K_t^\alpha L^{1-\alpha} = C_t + I_t$
Capital dynamics	$\mathrm{d}K_t = \left(\Phi\left(\frac{I_t}{K_t}\right) - \delta\right) K_t\mathrm{d}t$	$K_{t+1} = \left(\Phi\left(\frac{I_t}{K_t}\right) + (1-\delta)\right) K_t$
Habit dynamics	$\mathrm{d}X_t = (bC_t - aX_t)\,\mathrm{d}t$	$X_{t+1} = \tilde{b}C_{t-1} + \tilde{a}X_{t-1}$
TFP dynamics	$\mathrm{d}A_t = -\rho_A A_t\mathrm{d}t + \sigma_A\mathrm{d}B_{A,t}$	$A_{t+1} = \tilde{\rho}_A A_t + \tilde{\sigma}_A\epsilon_{A,t+1}$
TFP shock	$(B_{A,t+\Delta} - B_{A,t}) \sim N(0,\Delta)$	$\epsilon_{A,t} \sim N(0,1)$

Table C1. Summary of the two modeling frameworks. The table summarizes the two modeling frameworks in continuous and discrete time.

state is fully characterized by

$$\overline{A} = 0 \tag{C.16}$$

$$\overline{K} = \left[\frac{\alpha\exp(\overline{A})}{\rho+\delta}\right]^{\frac{1}{1-\alpha}} \tag{C.17}$$

$$\overline{C} = \exp(\overline{A})\overline{K}^\alpha - \delta\overline{K} \tag{C.18}$$

$$\overline{X} = \frac{b}{a}\overline{C} \tag{C.19}$$

$$\beta\overline{\overline{V}}_X = -\frac{1}{\rho+a}\left(\overline{C}-\overline{X}\right)^{-\gamma} \tag{C.20}$$

$$\beta\overline{\overline{V}}_K = \left(1-\frac{b}{\rho+a}\right)\left(\overline{C}-\overline{X}\right)^{-\gamma}, \tag{C.21}$$

where $\overline{\overline{V}}_X$ and $\overline{\overline{V}}_K$ denote the deterministic steady-state values of the costate variables for the capital stock and the habit formation in the discrete-time economy. By setting $\beta = 1/(1+\rho)$, and $\tilde{b} = b$ and $\tilde{a} = a$, we ensure that the steady state values of the capital stock and the long-run habit-to-consumption ratio are equal in the discrete- and continuous-time models.

C.3 Perturbation method

The equilibrium conditions of the model are summarized by equations (C.11)–(C.15). As in the continuous-time case, the policy functions that solve these conditions are not available in closed form and therefore will be approximated using perturbation methods.

As before, let the augmented stochastic process for the TFP be given by

$$A_{t+1} = \tilde{\rho}_A A_t + \eta \tilde{\sigma}_A \epsilon_{A,t+1},$$

where η is the perturbation parameter that controls the standard deviation of TFP shocks (not the variance as in the continuous-time model).

Following Schmitt-Grohe and Uribe (2004), the equilibrium conditions can be compactly written as

$$\mathbb{E}_t \left[\mathcal{H}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t; \eta) \right] = \mathbf{0}, \tag{C.22}$$

where $\mathbf{x}_t = [K_t, X_t, A_t]^\intercal$ is the vector of state variables at time t, with initial value $\mathbf{x}_0 > \mathbf{0}$, $\mathbf{y}_t = \left[\widetilde{V}_{K,t}, \widetilde{V}_{X,t}, \widetilde{V}_{A,t}, C_t \right]^\intercal$ is the vector of control variables at time t, and $\mathcal{H}$ is an operator that collects the equilibrium conditions (C.11)$-$(C.15). The deterministic steady state is then defined as the pair $(\overline{\mathbf{y}}, \overline{\mathbf{x}})$ that solves

$$\mathcal{H}(\overline{\mathbf{y}}, \overline{\mathbf{y}}, \overline{\mathbf{x}}, \overline{\mathbf{x}}; 0) = \mathbf{0}. \tag{C.23}$$

The solution to the discrete-time model in (C.22) takes the form

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t; \eta) \tag{C.24}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t; \eta) + \eta \tilde{\sigma}_A \epsilon_{A,t+1}, \tag{C.25}$$

where $\mathbf{g}(\cdot)$ is a vector of unknown policy functions that maps every possible value of $\mathbf{x}_t$ into $\mathbf{y}_t$, and $\mathbf{h}(\cdot)$ is a vector of unknown policy functions that maps every possible value of $\mathbf{x}_t$ into $\mathbf{x}_{t+1}$. Substituting into the functional operator that defines the equilibrium delivers the new operator

$$F(\mathbf{x}_t; \eta) \equiv \mathbb{E}_t \left[\mathcal{H} \left(\mathbf{g}(\mathbf{h}(\mathbf{x}_t; \eta) + \eta \tilde{\sigma}_A \epsilon_{A,t+1}; \eta), \mathbf{g}(\mathbf{x}_t; \eta), \mathbf{h}(\mathbf{x}_t; \eta) + \eta \tilde{\sigma}_A \epsilon_{A,t+1}, \mathbf{x}_t; \eta \right) \right] = \mathbf{0}. \tag{C.26}$$

A perturbation-based approximation to the solution of problem (C.22) builds a Taylor series expansion of the unknown policy functions around the deterministic steady state using the fact that (C.26) holds for any values of $\mathbf{x}_t$ and η. An immediate consequence of the latter is that all the partial derivatives of the functional $F(\mathbf{x}_t; \eta)$ must be zero, i.e.,

$$F_{x_i^k \eta^j}(\mathbf{x}_t; \eta) = 0, \quad \forall x, \eta, i, k, j,$$

where $F_{x_i^k \eta^j}(\mathbf{x}_t; \eta)$ denotes the derivative of F with respect to the i-th element in $\mathbf{x}_t$ taken k times, and with respect to η taken j times evaluated at $(\mathbf{x}_t; \eta)$.

A first-order approximation to the policy functions is given by

$$\begin{aligned}\mathbf{g}(\mathbf{x}_t;\eta) &\approx \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_\eta(\overline{\mathbf{x}};0)\eta\\ \mathbf{h}(\mathbf{x}_t;\eta) &\approx \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_\eta(\overline{\mathbf{x}};0)\eta,\end{aligned}$$

where $\mathbf{g}(\overline{\mathbf{x}};0)$ and $\mathbf{h}(\overline{\mathbf{x}};0)$ correspond to the deterministic steady-state values of the control and state variables derived from (C.23), and where the constants $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)$, $\mathbf{g}_\eta(\overline{\mathbf{x}};0)$, $\mathbf{h}_\eta(\overline{\mathbf{x}};0)$ can be determined by solving the system of equations formed by

$$\begin{aligned}F_{x_i}(\overline{\mathbf{x}};0) &= 0, \quad \forall i\\ F_\eta(\overline{\mathbf{x}};0) &= 0.\end{aligned}$$

We refer to the first set of equations (those not involving the perturbation parameter) as the perfect-foresight component of the approximation, and to the second set of equations as the stochastic component of the approximation (cf. Andreasen and Kronborg, 2018).

The system of equations resulting from the perfect-foresight component is quadratic in the unknowns $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)$. We pick the solution that ensures stability of the model's endogenous variables, i.e., the stable manifold (e.g., Blanchard and Kahn, 1980; Klein, 2000). The remaining constants, $\mathbf{g}_\eta(\overline{\mathbf{x}};0)$ and $\mathbf{h}_\eta(\overline{\mathbf{x}};0)$, correspond to the solution of the system of equations formed by the stochastic component, the unique solution being $\mathbf{g}_\eta(\overline{\mathbf{x}};0)=\mathbf{h}_\eta(\overline{\mathbf{x}};0)=0$ (cf. Fernandez-Villaverde et al., 2016). Hence

$$\mathbf{g}(\mathbf{x}_t;\eta) \approx \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\bar{\mathbf{x}}) \tag{C.27}$$

$$\mathbf{h}(\mathbf{x}_t;\eta) \approx \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\bar{\mathbf{x}}), \tag{C.28}$$

implying that up to a first order, the approximation exhibits certainty equivalence, i.e., the solution of the model is identical to the solution of the same model in the absence of uncertainty, $\eta=0$.

Similarly, a second-order approximation to the policy functions is given by

$$\begin{aligned}\mathbf{g}(\mathbf{x}_t;\eta) \approx{}& \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_\eta(\overline{\mathbf{x}};0)\eta\\ &+\tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes\eta+\tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)\eta^2\end{aligned}$$

and

$$\begin{aligned}\mathbf{h}(\mathbf{x}_t;\eta) \approx{}& \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_\eta(\overline{\mathbf{x}};0)\eta\\ &+\tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})\otimes\eta+\tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)\eta^2,\end{aligned}$$

where the definition of the matrices $\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$, $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$, and $\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)$ can be found in Binning (2013). These unknown coefficients correspond to the solution of the system of equations formed by

$$\begin{aligned} F_{x_i x_j}(\overline{\mathbf{x}};0) &= 0 \quad \forall i,j, \\ F_{\eta\eta}(\overline{\mathbf{x}};0) &= 0. \end{aligned}$$

As shown in Schmitt-Grohe and Uribe (2004), the cross derivatives $\mathbf{g}_{\mathbf{x}\eta}$ and $\mathbf{h}_{\mathbf{x}\eta}$ evaluated at $(\overline{\mathbf{x}};0)$ are zero, and hence the second-order perturbation reduces to

$$\mathbf{g}(\mathbf{x}_t;\eta) \approx \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0) \quad \text{(C.29)}$$

$$\mathbf{h}(\mathbf{x}_t;\eta) \approx \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0). \quad \text{(C.30)}$$

Hence, solving a second-order approximation introduces a constant correction in the policy functions that account for the effects of risk given by $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)$, while the slopes of the policy functions are not affected by risk as $\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0) = \mathbf{h}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0) = 0$.

C.4 Calibration

For the numerical exercises presented in the paper we calibrate the discrete-time model as in the continuous-time case. In particular, we set the risk aversion parameter and the share of capital income to $\gamma = 2$ and $\alpha = 0.36$, respectively. The annual values for the subjective discount rate and the depreciation rate are fixed to $\beta = 1/(1+\rho) = 0.9606$ and $\delta = 0.0963$, respectively. For the habit process we use $a = 1$ and $b = 0.82$, while the adjustment cost parameter is calibrated to $\xi = 0.3261$. Finally, following Christensen et al. (2016), the annual values for the persistence and volatility of the TFP are set to $\tilde{\rho}_A = 0.8145$ and $\tilde{\sigma}_A = 0.0278$, respectively.

C.5 Risky steady state

Following de Groot (2013), it is possible to approximate the risky steady state of a discrete-time economy by making use of the second-order approximation around the deterministic steady state. First, consider the second-order approximation to the transition equation for the state variables in (C.30)

$$\mathbf{x}_{t+1} = \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0) + \tilde{\sigma}_A \epsilon_{A,t+1}.$$

By setting the random disturbances to zero, $\epsilon_{A,t+1} = 0$, we compute the risky steady-state value of the state variables as the vector $\hat{\mathbf{x}}$ that satisfies $\mathbf{x}_{t+1} = \mathbf{x}_t = \hat{\mathbf{x}}$, and thus that solves the quadratic equation

$$\hat{\mathbf{x}} = \overline{\mathbf{x}} + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) \otimes (\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0).$$

Once $\hat{\mathbf{x}}$ is computed, it is possible to back out the implied risky steady-state value for the control variables, $\hat{\mathbf{y}}$, by simply inserting $\hat{\mathbf{x}}$ into (C.29)

$$\hat{\mathbf{y}} = \overline{\mathbf{y}} + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\hat{\mathbf{x}} - \overline{\mathbf{x}}) \otimes (\hat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0).$$

The corresponding risky steady state values for habit, capital stock, and consumption resulting from the calibration in Section C.4 are $\hat{X} = 1.0608$, $\hat{K} = 4.7184$, and $\hat{C} = 1.2936$, respectively.

D Policy and Impulse-Response functions

For comparison purposes, this appendix reports the policy and impulse-response functions obtained from the discrete-time model in Appendix C. They are computed using the software platform `dynare`.

Figure D1 compares approximated policy functions for consumption across orders of approximation; on the left-hand side (LHS) for the continuous-time case and on the right-hand side (RHS) for the discrete-time case. Note that our calibration implies identical deterministic steady states across time assumptions. The policy function for consumption approximated by means of a first-order perturbation in the discrete-time model (solid line on the RHS) goes through the deterministic steady state (approximation point) which suggests that the approximation is certainty equivalent. In contrast, the First-Order approximation of the policy function in the continuous-time model (solid line on the LHS) does not go through the deterministic steady state (approximation point) indicating that it is not certainty equivalent. Only by shutting down the risk-correction, $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0) = 0$, the continuous-time model First-Order CE approximation (dotted line on the LHS) will go through the deterministic steady state. Hence, as claimed in the main text, the First-Order CE resembles the first-order approximation in discrete time. Further note that in continuous time the First-Order delivers an approximation that is close to that provided by the Second-Order approximation (dashed line on the LHS) in the neighborhood of the deterministic steady state. The same does not occur in discrete time.

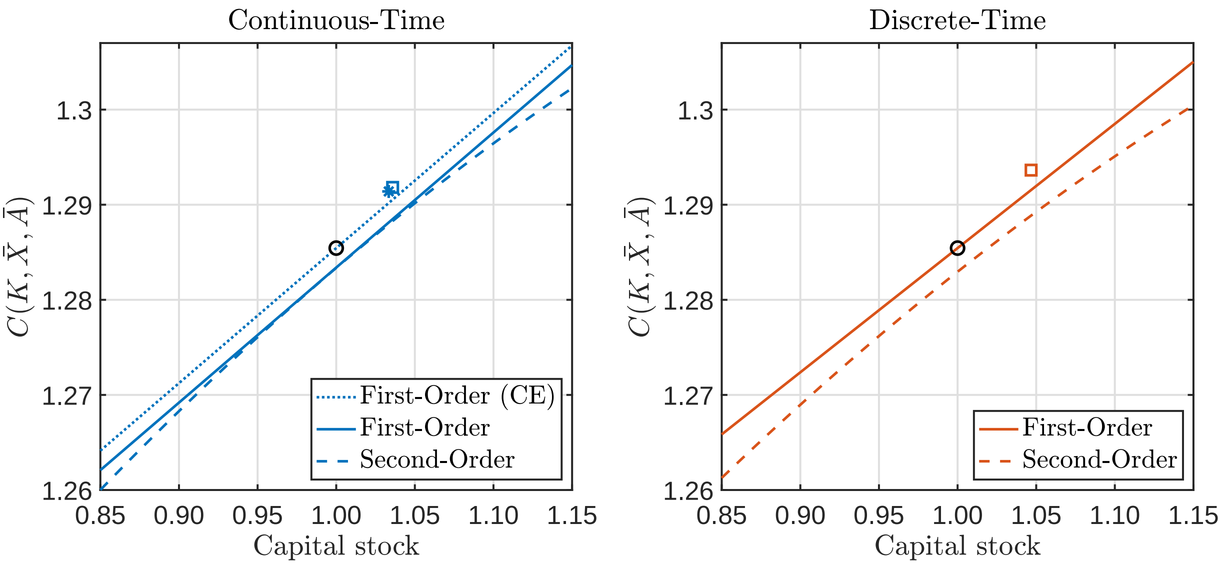

Figure D1. Continuous- and discrete-time approximated policy functions: First- and second-order approximations of the policy function for consumption around the deterministic steady state along the capital lattice while keeping habit and productivity at their deterministic steady-state values, $C\left(K, \overline{X}, \overline{A}\right)$. A circle denotes the deterministic steady state, a star denotes the first-order approximation and a square the second-order approximation of the risky steady state.

Figure D2 plots the approximated IRFs for consumption to a one standard deviation[1] shock in TFP across orders of approximation: on the LHS the continuous-time case and on the RHS the discrete-time case. As the first-order approximation in discrete time (solid line on the RHS) is certainty equivalent, the corresponding IRF starts in the deterministic steady state, where it also converges to. Comparing this IRF to the IRF from the First-Order CE in continuous time (dotted line on the LHS), one concludes that they are similar. Further, note that on the RHS we observe a large difference between first- and second-order approximated IRFs (solid vs. dashed line), since in the discrete time case only a second-order approximation provides risk-correction. In contrast, the differences between the IRFs resulting from the First- and Second-Order approximation (solid vs. dashed line on the LHS) are minor in the continuous time case, which reflects the fact that both approximations of the policy function are similar in the neighborhood of the deterministic steady state (see Figure D1). All these considerations suggest that the main weakness of the first-order approximation in discrete time is not that it is linear, but rather that it is certainty equivalent. Therefore, the continuous-time First-Order approximation is especially useful in situations in which risk matters but nonlinearities are negligible.

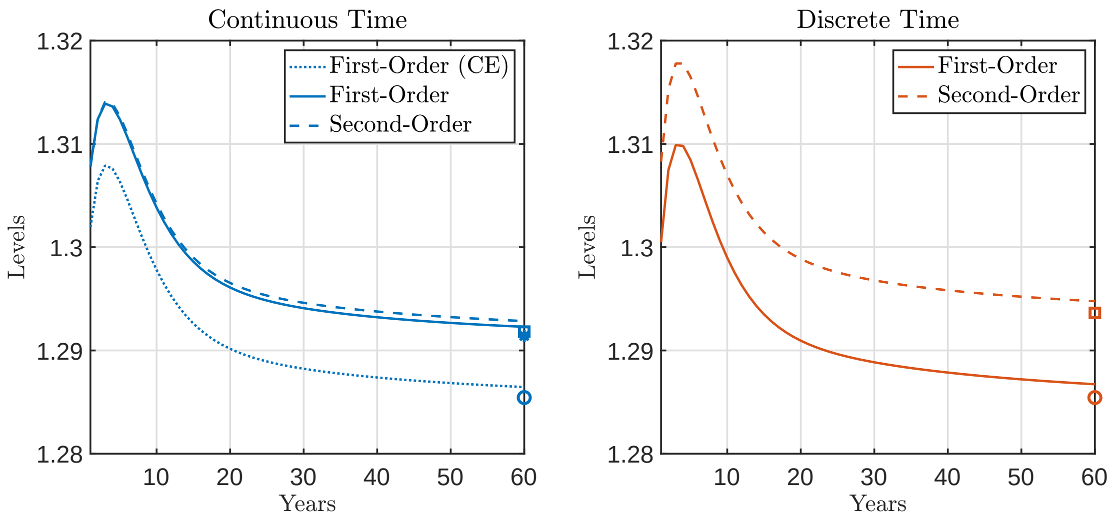

Figure D2. Impulse-Response function to a one s.d. shock in TFP (Discrete-time model): It plots the impulse response functions (IRFs) for the levels of aggregate consumption, capital, and habit when time is discrete. The variables in the economy are assumed to be in their corresponding risky steady states before the shock hits. A circle denotes the deterministic steady state, a star denotes the first-order approximation and a square the second-order approximation of the risky steady state.

[1]More precisely, for ease of comparison, we impose in both time assumptions an impulse of one standard deviation of the continuous-time model, i.e. $\sigma_A = 0.0307$.

E Pricing errors

Figure E1 reports the percentage (absolute) pricing errors for different approximations under the assumption that the true data generating process is given by the global approximation to the nonlinear stochastic model. The First-Order (CE), First-Order and Second-Order have been already introduced in Figure 3 in the main text.

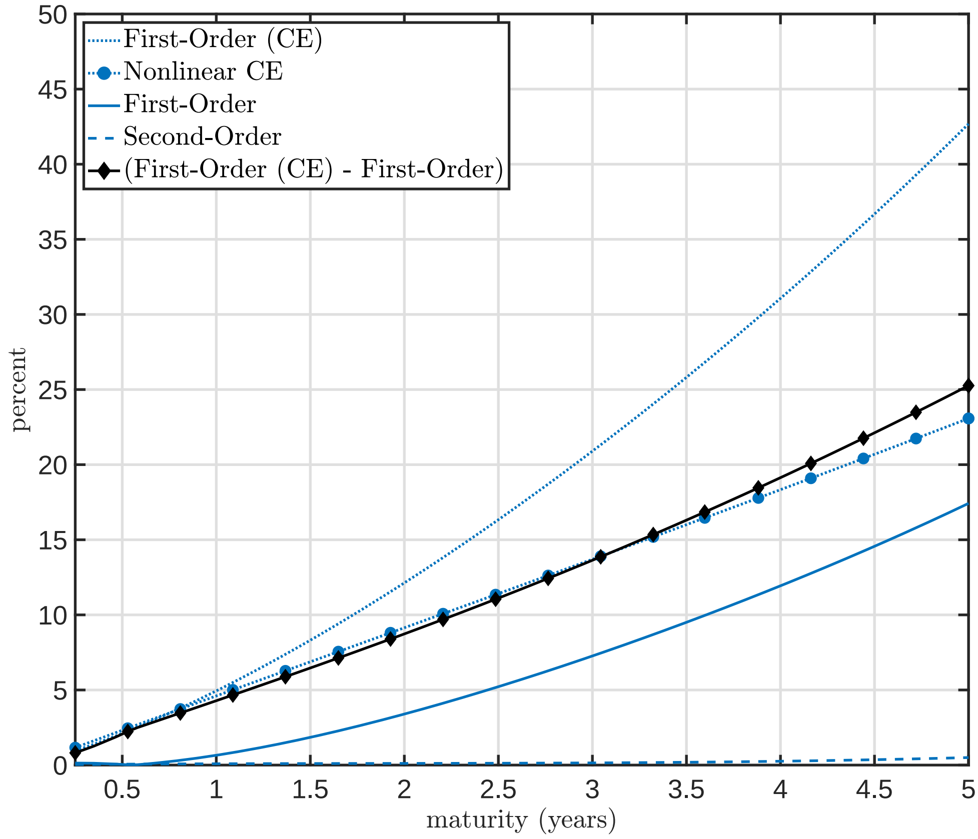

Figure E1. Decomposition of pricing errors: The graph plots the pricing errors resulting from First-Order (CE), First-Order, Nonlinear (CE), Second-Order, and the difference between the first two assuming that the true data generating process is the nonlinear stochastic solution.

Recall that the pricing error generated by the First-Order (CE) can be decomposed into: (i) the error stemming from the linearization of the nonlinear and stochastic policy function, which is captured by the First-Order approximation, and (ii) the error stemming from the imposition of certainty equivalence in the linear world. The latter is captured by the difference between First-Order (CE) and First-Order, and it is represented by the black line with diamonds: it measures the fraction of the pricing error that can be attributed to the imposition of certainty equivalence when using the First-Order (CE) solution. This measure can alternatively be interpreted as the reduction in the pricing errors that will be induced by the use of the (risk-sensitive) First-Order approximation.

Figure E1 presents an additional breakdown of the pricing errors generated by the use of the First-Order (CE). In particular, it is possible to decompose this error into: (i) the error stemming from imposing certainty equivalence in the nonlinear world, and

(ii) the error stemming from linearization in the presence of certainty equivalence. The former is given by the approximation of the policy function using a global method in a deterministic environment (Nonlinear CE, blue line with circles), while the latter would be given by the difference between the Nonlinear CE and the First-Order (CE).

By comparing the black line with diamonds and the blue line with circles, we can infer the effects from imposing certainty equivalence on the quality of the approximation. The first one provides a measure of this error in the linearized world, while the second one does it in the nonlinear world. The results suggest that the error reduction one would obtain from using the First-Order approximation is very close to the error one makes when imposing certainty equivalence in the nonlinear global solution. This can be interpreted as our First-Order approximation removing all of the error stemming from certainty equivalence such that all the remaining error can be attributed to linearization and, thus, is inevitable. Therefore, the First-Order approximation in continuous time makes it possible to account for the effects of risk in a linear framework.

References

Andreasen, M. M. and A. Kronborg (2018): "The Extended Perturbation Method," Tech. rep.

Binning, A. (2013): "Solving second and third-order approximations to DSGE models: A recursive Sylvester equation solution," Working Paper 2013/18, Norges Bank.

Blanchard, O. J. and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 1305–1311.

Christensen, B. J., O. Posch, and M. van der Wel (2016): "Estimating Dynamic Equilibrium Models Using Mixed Frequency Macro and Financial Data," *Journal of Econometrics*, 194, 116 – 137.

de Groot, O. (2013): "Computing the Risky Steady State of DSGE Models," *Economics Letters*, 120, 566–569.

Fernandez-Villaverde, J., J. Rubio-Ramirez, and F. Schorfheide (2016): "Solution and Estimation Methods for DSGE Models," in *Handbook of Macroeconomics*, ed. by J. B. Taylor and H. Uhlig, Elsevier, vol. 2 of *Handbook of Macroeconomics*, chap. 9, 527 – 724.

Grishchenko, O. V. (2010): "Internal vs. External Habit Formation: The Relative Importance for Asset Pricing," *Journal of Economics and Business*, 62, 176–194.

Jermann, U. J. (1998): "Asset Pricing in Production Economies," *Journal of Monetary Economics*, 41, 257–275.

Klein, P. (2000): "Using the Generalized Schur Form to a Solve Multivariate Linear Rational Expectations Model," *Journal of Economic Dynamics and Control*, 24, 1405–1423.

Schmitt-Grohe, S. and M. Uribe (2004): "Solving Dynamic General Equilibrium Models Using a Second-order Approximation to the Policy Function," *Journal of Economic Dynamics and Control*, 28, 755–775.