Optimal control of investment, premium and deductible for a non-life insurance company

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July 22, 2021

Abstract

A risk-averse insurance company controls its reserve, modelled as a perturbed Cramér-Lundberg process, by choice of both the premium p and the deductible K offered to potential customers. The surplus is allocated to financial investment in a riskless and a basket of risky assets potentially correlating with the insurance risks and thus serving as a partial hedge against these. Assuming customers differ in riskiness, increasing p or K reduces the number of customers $n(p, K)$ and increases the arrival rate of claims per customer $\lambda(p, K)$ through adverse selection, with a combined negative effect on the aggregate arrival rate $n(p, K)\lambda(p, K)$. We derive the optimal premium rate, deductible, investment strategy, and dividend payout rate (consumption by the owner-manager) maximizing expected discounted lifetime utility of intermediate consumption under the assumption of constant absolute risk aversion. Closed-form solutions are provided under specific assumptions on the distributions of size and frequency claims.

JEL classification: G11, G22, C60, D82

Keywords: Stochastic optimal control, Hamilton-Jacobi-Bellman equation, Jump-diffusion, Adverse selection, Premium control, Deductible control, Optimal investment strategy

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1 Introduction

The optimal choice by an insurance company of strategies for dividend payout, investment in safe and risky financial assets, and reinsurance arrangements have been studied heavily by means of stochastic control, see, e.g., Schmidli (2008). A further control variable at the company's disposal is the premium. In the non-life insurance literature, this is typically set as the net premium, i.e., arrival rate times expected size of claims, with an added safety loading based on the expected value principle, standard deviation principle, variance principle, or similar. As an alternative to these premium principles, Asmussen et al. (2013) consider the premium as a direct control, chosen optimally to balance the trade-off between resulting portfolio size and average profit per customer. Analysis of the individual customer's decision problem of whether or not to insure at a given premium delivers both the dependence of portfolio size on premium and, as in Rothschild and Stiglitz (1976), a further adverse selection effect on the riskiness of the average customer. Thus, while a higher premium increases revenue per customer and reduces portfolio size, it leads in addition to a higher average claim arrival rate. Thøgersen (2016) extends this analysis by investigating the dependence of the optimal premium on a fixed deductible, taken as given rather than controlled by the company. In the present paper, we consider the combined stochastic control problem faced by an insurance company simultaneously choosing premium, deductible, safe as well as risky financial investments, and the resulting residual dividend payout or consumption flow. The optimization criterion considered is the expected discounted lifetime utility from this latter flow. Our work provides a unifying framework, combining and generalizing both the literature on optimal investment of the insurance company's surplus and the optimal premium control literature.

We consider an infinitely-lived non-life insurance company that collects premiums from policyholders in continuous time, and covers net (of deductible) claims arriving according to a compound Poisson process. The contract offered is characterized by the premium rate, $p_t > 0$, and the deductible, $K_t \geq 0$. The company decides on both p_t and K_t to control its exposure to the insurance risk stemming from its contractual obligations, taking into account the effects on the size of the insurance portfolio, $n(p_t, K_t)$, and the average arrival rate of claims, $\lambda(p_t, K_t)$. Further, the company has access to a risk-free asset and a basket of risky assets for investment of the surplus from its insurance activities. Thus, it can enhance its risk management by partially hedging its insurance risk exposure through a financial portfolio that exploits any correlation between unexpected variations in surplus from the insurance business and returns to assets in the investment opportunity set. Correlation between the company's assets and liabilities can arise due to common dependence on the economic and natural environment, including systemic risk and contagion between the financial and insurance sectors. Despite the resulting hedging opportunity, the market remains incomplete, since the risk arising from insurance claims cannot be completely eliminated under any investment strategy. The main analysis of the financial market side is cast in the setting of a standard Black and Scholes (1973) specification in which all financial assets are driven by Brownian motions and exhibit constant expected rates of return, volatilities, and correlations through time, in order to focus on the implications of the financial investments for the insurance side and the finance-insurance interaction. In addition, we briefly outline the steps involved in extensions of the specification to accommodate state-dependent asset dynamics, following Merton (1971), and jumps in asset prices, following Merton (1976) and others.

An early contribution to the premium control literature was by Martin-Löf (1983), who proposed a linear feedback rule, setting the premium as a linear function of the surplus (reserve) and expected claims. Vandebroek and Dhaene (1990) showed that this is close to the optimal policy in a linear-quadratic control problem with the premium as control and the surplus as state. These papers did not consider the dependence of portfolio size and claim arrival rate on the choice of premium, and the linear nature of the premium policy stemmed from an assumption that the quadratic objective penalized deviations in premiums and surplus levels from otherwise unspecified target levels assumed to derive from ruin theory. Taylor (1986) adopted a demand function specification for portfolio size as function of the premium, with price elasticity depending on average (or market) premium. Several subsequent studies have extended the deterministic discrete-time framework of Taylor. For example, Emms et al. (2007) consider a stochastic continuous-time generalization, and Pantelous and Passalidou (2017) consider a stochastic demand function (portfolio size) in discrete time. However, none of these papers allows for adverse selection effects of changes in premium on the riskiness of customers. Højgaard (2002) allows the claim arrival rate or intensity to depend in an unspecified manner on a safety loading, and hence the premium, assuming the expected value principle. In effect, the optimization is with respect to the arrival rate, but this is assumed proportional to portfolio size, and not separated from the average arrival rate per customer, i.e., adverse selection is still not accommodated. Joint determination of mutually consistent portfolio size and arrival rate requires analysis of the customer's problem, with endogenous derivation of (rather than assumption about) the demand function and average customer risk, following Asmussen et al. (2013), who assume no deductible, and the extension to a fixed deductible in Thøgersen (2016). In the insurance industry, a deductible may be introduced for a variety of reasons (see, e.g., Burnecki et al., 2005). First, it avoids handling costs associated with a large number of small claims; second, it prevents customers from making claims by sharing the cost of claims with the company; third, it allows the company to reduce the premium rate. We focus on a further strategic role of the deductible, namely, the insurance company can control the size and risk composition of its portfolio through the choice of (p_t, K_t), jointly. Thus, we contribute to the optimal premium control literature by adding optimal deductible control, as well as financial asset investment and hedging opportunities.

The second strand of literature that we build on is that on optimal investment in safe and risky assets of an insurance company's surplus. Browne (1995) allowed correlation between the financial and insurance risks, as in our case, and derived the optimal portfolio strategy of an insurance company taking the premium rate as given, and facing an insurance risk process described by the diffusion approximation to the classical Cramér-Lundberg model. Hipp and Plum (2000) and Yang and Zhang (2005) studied the corresponding asset allocation problems in which the surplus from the insurance business is instead modelled by a compound Poisson or jump-diffusion process. Using dynamic programming techniques, they obtained closed-form expressions for the optimal investment policy. Zheng et al. (2016) consider robust optimal portfolio and reinsurance arrangements for an ambiguity-averse insurance company. Zhou et al. (2017) consider optimal investment and premium control, but assume an unspecified monotone relation between safety loading and claim arrival rate, as in Højgaard (2002), i.e., no adverse selection, and they rely on a diffusion approximation to the surplus. With the exception of this paper, the investment literature takes the premium as given, and none of the papers considers a deductible. We contribute to the investment literature by adding the premium and deductible as further controls, including the resulting adverse selection effects, and recognizing the compound Poisson component of the surplus process.

We assume that the surplus is driven by a perturbed Cramér-Lundberg process that in addition to the premium flow and compound Poisson process includes a diffusion component, following Gerber (1970). This perturbation captures any uncertainty about premium income and additional (beyond the compound Poisson process) uncertainty about aggregate claims (see Dufresne and Gerber, 1991, Furrer and Schmidli, 1994, and Schmidli, 1995), possibly stemming from variations in administrative expenses, operational costs, economic environment, or business conditions more generally. Specifically, we allow for correlation between the insurance and financial risk components of the company's balance sheet through this perturbation term. There are good reasons to expect such correlation to matter for insurance companies. Besides general co-movements of administrative and operational costs with the state of the economy and thus asset prices through the business cycle, correlation between insurance claims and equities can arise due to the occurrence or anticipation of natural disasters, epidemics, financial crises, economic recession, etc. Examples abound, e.g., Achleitner et al. (2002) consider the September 11, 2001, attack on the World Trade Center and document strong correlation impacts on the relation between underwriting and investment risks faced by property and casualty insurers, stemming from the simultaneous shock to their assets and liabilities. During the financial crisis of 2008, claims covered by credit insurance increased dramatically, while financial markets tumbled. Global warming and climate change are expected to increase the incidence of storms, droughts, floods, and thus property claims, and at the same time lead to economic and social instability, hence inducing contagion between insurance and financial markets (Hainaut, 2017). A currently relevant example is the ongoing coronavirus COVID-19 pandemic, which is generating massive claims from travel insurance, event cancellation, and health policy holders, while simultaneously depressing hotel, restaurant, tourism, entertainment, transportation, manufacturing, and many other economic sectors, and thus asset prices. Ward and Zurbruegg (2000) document co-movement between the insurance sector and the aggregate economy, and Billio et al. (2012) provide statistical evidence of systemic risk and correlation between the insurance and financial sectors. To accommodate such correlation in our model, we assume that the prices of financial assets available to the insurance company are driven by a vector of geometric Brownian motions, along the lines of Merton (1971), that are potentially correlated with the insurance surplus through the perturbation term, and we investigate the risk management implications of such correlation.

The insurance company's objective function deserves some mention. We consider expected lifetime utility from the running consumption or dividend flow extracted from the insurance surplus and financial wealth process, discounted at a constant rate $\delta > 0$, representing the subjective rate of time preference or impatience. This reflects the idea that the company pays dividends to a risk averse owner, who then consumes the running dividend, and that the company acts in the owner's interest, including the financial investment decisions. An alternative would be for the company to maximize expected discounted dividends, without regard to risk aversion, and let the owner handle the optimization of the consumption path, using the financial markets to smooth the dividend stream. For example, in the diffusion case, with dividends as the only control, Radner and Shepp (1996) and Asmussen and Taksar (1997) considered the maximization of expected dividends discounted at a constant rate. However, separation of company ownership and management actually requires that the company maximize shareholder value. With the insurance surplus correlated with financial markets, this is given by expected dividends discounted instead at a rate appropriately adjusted for systematic or market risk, as in the Merton (1973) intertemporal capital asset pricing model (CAPM), with a risk premium given by a quantity of risk, reflecting the correlation, multiplied by the market price of risk, reflecting the risk aversion of the representative agent. In the present case, the representative agent corresponds to the owner, and rather than looking for a beta-adjustment in the discount rate for dividends along these lines, we consider directly the optimization of the owner's expected discounted lifetime utility, thus collapsing ownership and control, with risk aversion reflected through the utility function rather than the discount rate.

We assume that the instantaneous utility function exhibits constant absolute risk aversion (CARA), following Merton (1969, 1971). In economics and finance, expected discounted CARA utility of running (intermediate) consumption has been used widely, e.g., in applications to precautionary saving and incomplete markets, see Kimball and Mankiw (1989), Caballero (1990), Svensson and Werner (1993), and Wang (2004, 2006, 2009). In insurance, alternative criterions have been considered, e.g., minimization of ruin probability in the premium control literature, Asmussen et al. (2013), and in the investment literature either maximization of CARA utility of terminal wealth (as opposed to running consumption), Yang and Zhang (2005), or minimization of ruin probability, Hipp and Plum (2000). Browne (1995) showed that maximization of CARA utility of terminal wealth is equivalent to minimization of ruin probability when there is no risk-free asset. We include both utility of intermediate consumption and a risk-free asset, so maximization of expected discounted lifetime utility may not coincide with minimization of ruin probability. Indeed, along the optimal path, wealth can turn negative (ruin), which in our model is financed by borrowing at the risk-free rate $r > 0$ and short positions in the risky assets. A transversality condition is included to ensure that the company can ultimately repay any debts, in expected discounted value terms. In principle, we could consider maximization of expected discounted utility until ruin. Instead, we focus on the infinite-horizon problem and the extension to joint premium, deductible, and risky investment controls. As a supplement, we provide an analysis of the resulting ruin probability along the optimal path, including some novel analytical results for specific distributions of size and frequency of claims. However, while borrowing (shorting) strategies can drive wealth negative, paying out positive dividends (consumption) for negative surplus (wealth) is strictly forbidden in practice. The probability of this occurring is obviously bounded above by the ruin probabilities provided, but this bound can be tightened, and we derive conditions (parametric restrictions) that almost surely eliminate the event.

Our analysis of ruin probability relies on Lundberg's inequality and the so-called *net profit condition*, which for a perturbed Cramér-Lundberg process holds whenever the drift is strictly greater than the aggregate claim arrival rate multiplied by the expected net claim per policy, see, e.g., Sections 5.3 - 5.4 of Rolski et al. (2009) or Appendix D of Schmidli (2008). Along the optimal path in our setting, the relevant condition (see (A.1) in Appendix A) holds automatically if $\delta \leq r$, i.e., when the company is relatively patient, and we use the latter sufficient condition to provide explicit bounds on the ruin probability in special cases. In economics, Hall (1978) uses the special case $\delta = r$ to derive his random walk model of consumption. The opposite condition, $\delta > r$, is occasionally employed to capture the phenomenon that a given decision maker is instead relatively impatient. Examples in the individual household consumption literature relate to the Flavin (1981) excess sensitivity puzzle that consumption tends to track current income, hence contradicting the Friedman (1957) permanent income hypothesis. To address the puzzle, the impatience assumption is adopted, e.g., in the framework of the buffer-stock saving model of Carroll (1997). In this literature, to prevent households from borrowing excessively, liquidity or borrowing constraints must be imposed, e.g., as in Deaton (1991). Similar cash-in-advance constraints are imposed in monetary economics, see Clower (1967) and Lucas and Stokey (1987). In contrast, in our case, assuming a relatively patient insurance company, excessive borrowing is instead prevented by a sufficiently high interest rate and the transversality condition.

Following Asmussen et al. (2013), we consider the possibility that potential customers are heterogeneous, and not equally risky to the insurance company. In particular, they exhibit different arrival rates of claims. In a complete information version of the model, the individual customer knows its own arrival rate. However, there is asymmetric information between the customer and the insurance company, and the latter only knows the distribution of arrival rates across customers. This feature of the approach accommodates adverse selection. In the sequel, we consider an incomplete or partial information extension of the model, in which individual customers are uncertain about their own arrival rates. Using these assumptions in the analysis of the customer's problem, we endogenously derive mutually consistent functional forms of $n(p_t, K_t)$ and $\lambda(p_t, K_t)$, which enter into the company's optimization problem.

We find that the optimal contract to be offered by the insurance company is characterized by a premium rate and a deductible that are inversely related and constant through time in our setting. Optimal investment in risky assets is similarly constant, and includes both a speculative component and a hedge against insurance risk. The optimal consumption (dividend) rate is not constant through time, but fluctuates with wealth in an affine fashion, with marginal propensity to consume out of current wealth given by the interest rate, and the level of consumption depending in addition on the discount rate and risk aversion of the company, as well as on the risk characteristics of potential customers. The optimal premium exceeds the expected net premium, which would correspond to standard insurance pricing without safety loading, or to the expected reservation premium of hypothetical risk-neutral customers. It even exceeds the expected reservation premium after accounting for the risk aversion of customers, even though this level is shown to replace the average or market premium of Taylor (1986) and Emms et al. (2007) in the demand function. The excess of the optimal premium beyond the risk-adjusted expected reservation premium depends on the relative valuation of net claims by company and customer, thus reflecting that the company is risk averse, too, and exerts market power to protect itself against risky customers, i.e., the adverse selection aspect. In the particular case of exponentially distributed arrival rates of claims, we provide a closed form expression for the optimal premium rate, as function of the optimal deductible, and a semi-closed form characterization of the optimal deductible. For exponentially distributed claim sizes, optimal premium and deductible are both given in closed form, as are optimal financial investments and dividends (consumption). For more general situations, we provide theoretical characterizations and numerical examples to illustrate the optimal controls. We investigate how the optimal premium and deductible interact with investment and consumption decisions. In a sensitivity analysis, we explore the impact on the four optimal controls of variations in the parameters governing systemic risk and contagion in our model, i.e., the correlation between the surplus from the insurance business and the financial market risks, and the degree of perturbation to the surplus process. For the special case of a fixed deductible, we provide a closed-form expression for the optimal premium, thus complementing the result by Thøgersen (2016) of existence for ruin probability minimization, without explicit solution for the optimal premium.

The paper is organized as follows. The optimization problem faced by the insurance company is introduced in Section 2. Section 3 studies the optimal premium, deductible, investment, and dividend strategies maximizing expected discounted utility of intermediate consumption. The closed-form solution to the company's Hamilton-Jacobi-Bellman equation is derived, and the conditions for the verification theorem guaranteeing the existence of an optimal strategy are established. Section 4 characterizes the optimal premium and deductible for the case of random arrival rates of claims, with complete information on the customer's side, and asymmetric information between customer and insurance company. Section 5 presents some results for the extension to partial information on the customer's side. Section 6 concludes. The analysis of ruin probabilities in our setting is relegated to Appendix A, the extension to jumps in asset prices to Appendix B, and the derivation of the insurance demand function and average arrival rate of claims across customers in the partial information case to Appendix C.

2 The model

Consider a company that at time $t = 0$ allocates an initial endowment $w_0 > 0$ between a risk-free asset and a bundle of risky assets, and starts an insurance business by selling insurance contracts. Thereafter, at each instant $t > 0$, the company (i) receives premiums from policyholders; (ii) pays to policyholders when claims occur; (iii) rebalances portfolio holdings by buying or (short-)selling units of its financial assets; and (iv) consumes part of its wealth. Such consumption may be viewed as dividend payments to the owner-manager. The company is therefore subject to insurance risk arising from the insurance policies written and market risk arising from its investment strategy. Throughout, we assume that the financial market is frictionless, and therefore all securities can be traded continuously over time without transaction costs or taxes.

For a mathematical formulation of the problem faced by the company, let $(\Omega, \mathbb{P}, \mathcal{F})$ be a complete probability space endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, representing the information available to the company at time t. There are $I \geq 1$ risky assets with price processes $S_t^i,\ i = 1, \ldots, I$. The price of the risk-free asset, or money market account, is denoted by S_t^0. For each $i = 0, 1, \ldots, I$, we denote by θ_t^i the number of units of asset i held by the company at time $t \geq 0$. Thus, the financial wealth of the company at time t is defined by the market value of the financial portfolio,

$$W_t^{\boldsymbol{\theta}} := \boldsymbol{\theta}_t \cdot \mathbf{S}_t = \sum_{i=0}^{I} \theta_t^i S_t^i \,, \quad t \geq 0 \,,$$

where $\mathbf{S}_t = \left(S_t^0, \ldots, S_t^I\right)^\top$ and $\boldsymbol{\theta}_t = (\theta_t^0, \ldots, \theta_t^I)^\top$. The cumulative gains/losses to the financial investment portfolio associated with trading strategy $\boldsymbol{\theta}_t$ are therefore given by

$$G_t^{\boldsymbol{\theta}} := \int_0^t \boldsymbol{\theta}_\tau \cdot \mathrm{d}\mathbf{S}_\tau \,, \quad t \geq 0 \,.$$

In addition, the company holds an insurance portfolio. At each instant $t \geq 0$, the portfolio includes as many policies as the company can attract in the insurance market, given the premium rate $p_t \geq 0$ and deductible $K_t \geq 0$. Let $n(p_t, K_t)$ represent the number of customers in the insurance portfolio from a total population of N potential customers. The arrival rate of claims from a typical insured individual is assumed to be a positive-valued measurable function of $p_t \geq 0$ and $K_t \geq 0$, denoted $\lambda(p_t, K_t)$. Hence, p_t and K_t are the quantities the company can use to control its exposure to insurance risk. As each of the $n(p_t, K_t)$ customers pays the premium p_t, the cumulative gains/losses to the insurance portfolio are given by the reserve or surplus process

$$R_t^{p,K} := \int_0^t n(p_s, K_s) p_s \, ds - X_t^{p,K} \,, \quad t \geq 0 \,, \tag{2.1}$$

where $X_t^{p,K}$ is the aggregate insurance risk process, reflecting customer claims, as specified below.

A strategy $(\boldsymbol{\theta}, p, K)$ is said to be *self-financing* if $w_0 + G_t^{\boldsymbol{\theta}} + R_t^{p,K} \geq W_t^{\boldsymbol{\theta}}$, for all $t \geq 0$. The company pays out any residual wealth $w_0 + G_t^{\boldsymbol{\theta}} + R_t^{p,K} - W_t^{\boldsymbol{\theta}} \geq 0$ as dividends, for consumption by the owner-manager. We say that a self-financing strategy $(\boldsymbol{\theta}, p, K)$ is *admissible* if the cumulative consumption process

$$C_t^{\boldsymbol{\theta},p,K} := w_0 + G_t^{\boldsymbol{\theta}} + R_t^{p,K} - W_t^{\boldsymbol{\theta}} \,, \quad t \geq 0 \,,$$

is differentiable with respect to $t \geq 0$. The process $c_t^{\boldsymbol{\theta},p,K} := \frac{\mathrm{d}}{\mathrm{d}t} C_t^{\boldsymbol{\theta},p,K}$ is the instantaneous consumption rate. Focusing on admissible strategies, the budget constraint of the company can be written in differential form as

$$\mathrm{d}W_t^{\boldsymbol{\theta},p,K} = \boldsymbol{\theta}_t \cdot \mathrm{d}\mathbf{S}_t + n(p_t, K_t) p_t \, \mathrm{d}t - \mathrm{d}X_t^{p,K} - c_t^{\boldsymbol{\theta},p,K} \, \mathrm{d}t \,, \quad W_0^{\boldsymbol{\theta},p,K} = w_0 \,. \tag{2.2}$$

Although cumulative consumption is non-negative for an admissible strategy, by the self-financing property, neither wealth nor the consumption rate need be so. Negative wealth reflects borrowing at the risk-free rate, $\theta_t^0 < 0$, and shorting of risky assets, $\theta_t^i < 0$. Negative consumption rate reflects capital injections by the owner-manager into the company.

We assume the following dynamics for the financial and insurance markets. First, on the financial side, the price processes of the i^{th} risky asset, S_t^i, and the risk-free asset, S_t^0, are governed by the standard Black and Scholes (1973) model,

$$\mathrm{d}S_t^i \;=\; S_t^i\Big[\mu^i\,\mathrm{d}t + \sum_{j=1}^{I}\sigma^{ij}\,\mathrm{d}B_t^j\Big]\,,\;\; S_0^i > 0\,,\quad i = 1,\ldots,I\,, \tag{2.3}$$

$$\mathrm{d}S_t^0 \;=\; rS_t^0\,\mathrm{d}t\,,\;\; S_0^0 = 1\,, \tag{2.4}$$

where $\mathbf{B}_t = (B_t^1,\ldots,B_t^I)^\top$ is an I-dimensional vector of independent standard Brownian motions with respect to $\mathbb{F}$, $r > 0$ is the continuously compounded risk-free rate, $\boldsymbol{\mu} = (\mu^1,\ldots,\mu^I)^\top \in \mathbb{R}^I$ is the vector of expected returns to the risky assets, and $\boldsymbol{\sigma} = \big(\sigma^{ij}\big)_{1\le i,j\le I} \in \mathbb{R}^{I\times I}$ is the volatility or diffusion matrix, with associated variance-covariance matrix $\boldsymbol{\Sigma} := \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$.

Secondly, on the insurance side, given a progressively measurable process $(p,K) = \{(p_t,K_t)\}_{t\ge0}$ with values in $\mathbb{R}_+^2$, representing premium and deductible choices, the aggregate insurance risk process $X_t^{p,K}$ is given by the jump-diffusion process

$$\mathrm{d}X_t^{p,K} = n(p_t,K_t)b\,\mathrm{d}\overline{B}_t + \mathrm{d}J_t^{p,K}\,,\quad X_0 = 0\,, \tag{2.5}$$

with $(J_t^{p,K})_{t\ge0}$ the compound Poisson process

$$J_t^{p,K} = \sum_{m=1}^{N_t^{p,K}} (Y_m - K_{\tau_m})^+ \tag{2.6}$$

aggregating the net (of deductible) claims $(Y_m - K_{\tau_m})^+$ to be paid. Here, $(N_t^{p,K})_{t\ge0}$ is an adapted point process with intensity $n(p_t,K_t)\lambda(p_t,K_t)$ counting the total number of claims, τ_m are the jump times of $N^{p,K}$, and $Y_m \in (0,\infty)$ are i.i.d. random claim sizes with distribution $F(\mathrm{d}y)$, independent of $(N_t^{p,K})_{t\ge0}$. For concreteness, we focus in (2.6) on the case of a *fixed amount deductible*. Other types include franchise, proportional, limited proportional, and disappearing deductibles (see, e.g., Burnecki et al., 2005). The perturbation process $\overline{B}_t$ is a one-dimensional standard Brownian motion with respect to $\mathbb{F}$, accounting for additional uncertainty in the aggregate insurance risk process and assumed independent of the number and sizes of claims. The coefficient $b \ge 0$ in (2.5) captures the magnitude of the perturbation per insurance contract. Finally, $\langle B^i, \overline{B}\rangle_t = \rho^i t$, where $\rho^i \in (-1,1)$ is the correlation coefficient between the log-price of the i^{th} risky asset and the perturbation shocks to the aggregate insurance risk process.

Systemic or contagion risk corresponds to the perturbed case, $b > 0$, combined with negative insurance-finance correlation, $\rho^i < 0$, i.e., a positive shock to $\overline{B}$ (crisis, pandemic, etc.) reduces reserves, while a negative shock to B^i reduces the i^{th} asset price, for a loss on both the insurance and finance sides of the business. For modelling purposes, the perturbation process can be easily constructed as

$$\overline{B} := \boldsymbol{\rho}\cdot\mathbf{B} + \sqrt{1-\|\boldsymbol{\rho}\|^2}\,B^{I+1},$$

for $\boldsymbol{\rho} = (\rho^1,\ldots,\rho^I)^\top$ a correlation vector satisfying $\|\boldsymbol{\rho}\|^2 = \sum_{i=1}^{I}(\rho^i)^2 \le 1$, and B^{I+1} a one-dimensional standard Brownian motion with respect to $\mathbb{F}$, independent of $\mathbf{B}$. In the unperturbed case, $b = 0$, the definition in (2.5) reduces the company's insurance business surplus (2.1) to the classical Cramér-Lundberg process, at least for (p_s, K_s) constant. On the other hand, for $b > 0$, the company can construct investment strategies in the financial markets to manage the risks arising from the insurance activities. However, the latter cannot be completely eliminated, because market incompleteness remains, even for $\|\boldsymbol{\rho}\|^2 = 1$, due to the unspanned risks arising from the insurance claims. In this case, the modelling of the surplus corresponds to a perturbed Cramér-Lundberg process.

Remark 2.1. In an insurance portfolio with n outstanding (independent) contracts and zero deductible, the total number of claims up to time t is given by

$$\sum_{l=1}^{n} \sum_{m=1}^{N_t^l} Y_m^l \,,$$

where Y_m^l are i.i.d. random variables with distribution $F(\mathrm{d}y)$, and N^l independent Poisson processes with arrival rates λ^l, say, with N_t^l representing the number of claims by customer l up to time t. It is straightforward to show that this sum of independent compound Poisson processes has the same distribution as a compound Poisson process with arrival rate $n\lambda$ and claims with distribution $F(\mathrm{d}y)$, where $\lambda = (1/n)\sum_{l=1}^{n} \lambda^l$ is the average arrival rate across customers, see, e.g., Proposition 3.3.4 in Mikosch (2004). Since our goal is to maximize the expected value of a function of the wealth process, this equivalence in law justifies the use of the compound Poisson process with arrival rate $n\lambda$ as a model of total claims in the aggregate risk process. ■

For each $t \geq 0$, let $A_t^i = \theta_t^i S_t^i$ denote the amount of financial wealth invested in the i^{th} risky asset. The remaining financial wealth is invested in the risk-free asset. Using $\mathbf{A}_t = (A_t^1, \ldots, A_t^I)^\top$ as control variable, the budget constraint (2.2) transforms into the controlled stochastic differential equation (SDE)

$$\begin{aligned} &\mathrm{d}W_t^{c,p,K,\boldsymbol{A}} = [rW_t - c_t]\,\mathrm{d}t + \mathbf{A}_t^\top\left[(\boldsymbol{\mu} - r\underline{\mathbf{1}})\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t\right] \\ &+\, n\,(p_t, K_t)\left(p_t\,\mathrm{d}t - b\,\mathrm{d}\overline{B}_t\right) - \int_{\mathbb{R}_+} (y - K_t)^+ N^{p,K}(\mathrm{d}y, \mathrm{d}t)\,, \quad W_0^{c,p,K,\boldsymbol{A}} = w_0\,, \end{aligned} \tag{2.7}$$

where $\underline{\mathbf{1}} = (1, \ldots, 1)^\top \in \mathbb{R}^I$. Here, $N^{p,K}(\mathrm{d}y, \mathrm{d}t)$ denotes a (random) jump measure with compensator $n(p, K)\lambda(p, K)F(\mathrm{d}y)\,\mathrm{d}t$.

For a control policy described by the process $(c, p, K, \mathbf{A})$ with values in $\mathbb{R} \times \mathbb{R}_+^2 \times \mathbb{R}^I$ at time $t \geq 0$, we consider the expected discounted lifetime utility derived from consumption,

$$V(c, p, K, \mathbf{A}) := \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\,\mathrm{d}t \,\middle|\, W_0 = w_0\right], \tag{2.8}$$

subject to the budget constraint (2.7). Here, company (owner-manager) preferences are characterized by (δ, η), with $\delta > 0$ the discount rate, i.e., the subjective rate of time preference, or impatience, and $\eta = -U''/U' > 0$ the constant absolute risk aversion (CARA) coefficient, with $U(x) = -\frac{1}{\eta}e^{-\eta x}$ the instantaneous utility (see Pratt, 1964). This utility function plays a prominent role in insurance mathematics and actuarial practice, as it ensures that the principle of "zero utility" yields a fair premium, independent of the level of reserves (see Goovaerts et al., 1990, Section II.6).

The policy $(c, p, K, \mathbf{A})$ is *admissible* if it is adapted to $\mathbb{F}$ and (2.8) is strictly negative, for all initial wealth levels w_0. The reason for this definition is that $U < 0$, with $U(x) \to 0$ as $x \to \infty$.[1] Thus, although the company can borrow (or short risky assets) to render any (θ, p, K) self-financing, policies for which (2.8) is identically zero due to excessive consumption growth are ruled out. As we show, optimal $(p, K, \mathbf{A})$ are in fact constant through time, so there is no concern for these controls. In what follows, we denote the set of admissible policies by $\mathcal{A}$. The objective of the company is to find the policy that maximizes $V(c, p, K, \mathbf{A})$ over $\mathcal{A}$.

3 The Hamilton-Jacobi-Bellman equation

The optimal time-homogeneous value function is written as

$$\vartheta(w_0) = \sup_{(c,p,K,\mathbf{A}) \in \mathcal{A}} V(c, p, K, \mathbf{A}),$$

with $V(\cdot)$ given in (2.8). The associated transversality condition is

$$\lim_{T \to \infty} e^{-\delta T} \mathbb{E}[\vartheta(W_T)] = 0. \tag{3.1}$$

Intuitively, this requires that the company can ultimately repay any debts. Wealth can go negative, due to borrowing and short selling, and even diverge, but the transversality condition ensures that this does not occur in expected discounted value terms.

The problem is amenable to analysis using stochastic dynamic programming and the appropriate form of Itô's lemma for jump-diffusion processes, together with standard time-homogeneity arguments for optimal Markov policies in infinite-horizon problems. If $\vartheta(\cdot)$ is sufficiently differentiable and satisfies the transversality condition, then, for any level of financial wealth $w \in \mathbb{R}$, it satisfies the non-linear second-order integro-differential equation, usually referred to as the Hamilton-Jacobi-Bellman (HJB) equation (see Section III.7 of Fleming and Soner, 2006, and Sennewald, 2007),

$$-\delta\vartheta(w) + \sup_{(c,p,K,\mathbf{A}) \in \mathcal{A}} \left\{ U(c) + [\mathcal{L}^{c,p,K,\mathbf{A}}\vartheta](w) \right\} = 0, \tag{3.2}$$

where $\mathcal{L}^{c,p,K,\mathbf{A}}$ is the operator

$$\begin{aligned}[]
[\mathcal{L}^{c,p,K,\mathbf{A}}\vartheta](w) &= \left[rw + \mathbf{A}^\top(\boldsymbol{\mu} - r\underline{\mathbf{1}}) + n(p,K)p - c \right] \vartheta'(w) \\
&\quad + \frac{1}{2}\left[\left\| \boldsymbol{\sigma}^\top \mathbf{A} \right\|^2 + n(p,K)^2 b^2 - 2n(p,K) b \mathbf{A}^\top \boldsymbol{\sigma}\boldsymbol{\rho} \right] \vartheta''(w) \\
&\quad + n(p,K)\lambda(p,K) \left[\int_0^\infty \vartheta(w - (y-K)^+)\, F(\mathrm{d}y) - \vartheta(w) \right].
\end{aligned} \tag{3.3}$$

Assuming $F(\mathrm{d}y)$ is absolutely continuous, we conjecture that a solution to the HJB equation (3.2) takes the form

$$\vartheta(w) = \beta e^{-r\eta w}, \tag{3.4}$$

[1] In other cases, with $U(x) \to \infty$ as $x \to \infty$, the corresponding admissibility condition is $V < \infty$, for all w_0.

with $\beta < 0$. Finding an interior solution to the HJB equation for this conjecture requires maximizing

$$U(c) + cr\eta\beta e^{-r\eta w} \tag{3.5}$$

over $c \in \mathbb{R}$, and

$$\begin{aligned} r\eta[n(p,K)p + \mathbf{A}^\top(\boldsymbol{\mu} - r\underline{\mathbf{1}})] \\ - \frac{(r\eta)^2}{2}\Big[\left\|\boldsymbol{\sigma}^\top\mathbf{A}\right\|^2 + n(p,K)^2b^2 - 2n(p,K)b\mathbf{A}^\top\boldsymbol{\sigma}\boldsymbol{\rho}\Big] \\ - n(p,K)\lambda(p,K)q(K) \end{aligned} \tag{3.6}$$

over $(p, K, \mathbf{A}) \in \mathbb{R}^2_+ \times \mathbb{R}^I$, with

$$\begin{aligned} q(K) &:= \mathbb{E}\left[e^{r\eta(Y-K)^+}\right] - 1 \\ &= e^{-r\eta K}\int_K^\infty e^{r\eta y}\,F(\mathrm{d}y) - \bar{F}_Y(K) \\ &= e^{-r\eta K}\mathbb{E}\left[e^{r\eta Y}\,|Y \geq K\right]\mathbb{P}(Y \geq K) - \bar{F}_Y(K) \end{aligned} \tag{3.7}$$

representing a certain expected valuation of net claims $(Y - K)^+$, using the notation that $F(\mathrm{d}y)$ has density $f_Y(y)$, cumulative distribution function (c.d.f.) $F_Y(y) = \mathbb{P}(Y \leq y)$, and complementary c.d.f. $\bar{F}_Y(y) = 1 - F_Y(y)$. We assume throughout that $\mathbb{E}[e^{r\eta Y}] < \infty$, so that $q(K)$ is well defined for all $K \geq 0$. Note that (3.6) is independent of the level of wealth w and strictly concave in $\mathbf{A}$.

The first-order necessary optimality conditions for maximization with respect to c and $\mathbf{A}$ are, respectively,

$$e^{-\eta c} + r\eta\beta e^{-r\eta w} = 0\,, \tag{3.8}$$

$$\mu_i - r - r\eta([\boldsymbol{\sigma}\boldsymbol{\sigma}^\top\mathbf{A}]_i - n(p,K)b[\boldsymbol{\sigma}\boldsymbol{\rho}]_i) = 0\,, \quad i = 1, \ldots, I\,. \tag{3.9}$$

Solving (3.8) for c yields the candidate for optimal consumption $\hat{c}_t = \hat{c}(W_t)$,

$$\hat{c}(w) := rw - \frac{1}{\eta}\log(-\beta r\eta)\,, \tag{3.10}$$

with $\hat{c}(w)$ a stationary Markov control policy. Thus, optimal consumption is affine in wealth, reflecting interest income. Further, following Merton (1969), it is positive if and only if wealth is above a certain threshold, $W_t > w^\star$, with

$$w^\star = \frac{1}{r\eta}\log(-\beta r\eta)\,. \tag{3.11}$$

Optimal consumption may require borrowing or short selling, but this does not violate the self-financing property, which only requires non-negative cumulative consumption. For $W_t < w^\star$, consumption is negative, reflecting capital injections by the owner-manager, which may be required if wealth declines. Wealth can even turn negative, due to borrowing and short selling, but the transversality condition (3.1) implicitly imposes an asymptotic repayment constraint, and we do not rule out negative wealth (technical ruin). However, paying positive dividends (consumption) for negative surplus (wealth), corresponding to the event $\{w^\star < W_t < 0\}$, is strictly forbidden in practice. Obviously, the probability of this occurring is bounded above by the ruin probability implied by the chosen policy, and is zero if $w^\star \geq 0$.

Solving the system (3.9) yields candidates for optimal amounts of financial wealth to be invested in risky assets,

$$\hat{\mathbf{A}} = \mathbf{\Sigma}^{-1}\left[\frac{1}{r\eta}(\boldsymbol{\mu} - r\underline{\mathbf{1}}) + n(p,K)b\boldsymbol{\sigma}\boldsymbol{\rho}\right].$$

For $\boldsymbol{\sigma}$ invertible, this simplifies to

$$\hat{\mathbf{A}} = (\boldsymbol{\sigma}^{-1})^\top\left[\frac{1}{r\eta}\boldsymbol{\psi} + n(p,K)b\boldsymbol{\rho}\right], \tag{3.12}$$

where $\boldsymbol{\psi} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\underline{\mathbf{1}})$ is the I-vector of market prices of risk. Without systemic or contagion risk, i.e., if $b = 0$ or $\boldsymbol{\rho} = 0$, (3.12) reduces to the speculative or myopic demand for risky assets, cf. Merton (1969). In this case, there is complete separation between the financial and insurance markets, and thus the insurance risk is completely unspanned. In the perturbed case with insurance–finance correlation, $b > 0$ and $0 < \|\boldsymbol{\rho}\|^2 \leq 1$, optimal demand for risky assets involves an additional hedging component, the second term in (3.12), proportional to the size of the insurance portfolio, $n(p,K)$, as the company is using the financial markets to manage its exposure to insurance risk.

Substitution of the optimal investment strategy (3.12) into (3.6) produces the partially maximized or profile Hamiltonian criterion for (p,K),

$$\begin{aligned} Q(p,K) := \frac{1}{2}\|\boldsymbol{\psi}\|^2 + r\eta n(p,K)\left(b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\right) \\ - \frac{1}{2}\Big(r\eta b n(p,K)\Big)^2\left(1 - \|\boldsymbol{\rho}\|^2\right) - n(p,K)\lambda(p,K)q(K). \end{aligned} \tag{3.13}$$

Thus, the optimal premium p and deductible K are obtained as the interior solution to the maximization of (3.13) over $(p,K) \in \mathbb{R}_+^2$.

Proposition 3.1. *Suppose that the diffusion matrix $\boldsymbol{\sigma}$ is invertible and that there exists a pair of constants $(\hat{p},\hat{K})$ that maximizes $Q(p,K)$ in (3.13). Then the strategy $(\hat{c}(\cdot),\hat{p},\hat{K},\hat{\mathbf{A}})$ is optimal, with $\hat{\mathbf{A}}$ given by (3.12), also constant, and $\hat{c}(w)$ given by (3.10), with*

$$\hat{\beta} = -\frac{1}{r\eta}\exp\left(1 - \frac{1}{r}\left[\delta + Q(\hat{p},\hat{K})\right]\right) < 0.$$

Proof. The wealth process $\hat{W}$ controlled by the strategy $(\hat{c}(w),\hat{p},\hat{K},\hat{\mathbf{A}})$ is a jump-diffusion process with differential

$$\begin{aligned} \mathrm{d}\hat{W}_t = \frac{1}{\eta}\log(-\beta r\eta)\,\mathrm{d}t + \hat{\mathbf{A}}^\top[(\boldsymbol{\mu} - r\underline{\mathbf{1}})\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t] \\ + n(\hat{p},\hat{K})(\hat{p}\,\mathrm{d}t - b\,\mathrm{d}\overline{B}_t) - \int_{\mathbb{R}_+}(y - \hat{K})^+ N^{\hat{p},\hat{K}}(\mathrm{d}y,\mathrm{d}t). \end{aligned} \tag{3.14}$$

Note that $\mathbb{E}[\vartheta(\hat{W}_{t-})]$ is finite for all $t \geq 0$ since $\mathbb{E}[e^{-r\eta Y}] \leq 1$. Applying Itô's lemma, compensating the integrals with respect to $N^{\hat{p},\hat{K}}(\mathrm{d}y,\mathrm{d}t)$, and taking expected values, we get that $f(t) := \mathbb{E}[\vartheta(\hat{W}_{t-})]$ satisfies the ordinary differential equation (ODE)

$$f'(t) = -\left[r\log(-r\eta\beta) + Q(\hat{p},\hat{K})\right]f(t), \quad f(0) = \vartheta(w_0).$$

That is, $\mathbb{E}[\vartheta(\hat{W}_{t-})] = \vartheta(w_0)\exp(-[r\log(-r\eta\beta)+Q(\hat{p},\hat{K})]t)$. Inserting the guess (3.4) together with the strategy $(\hat{c}(w), \hat{p}, \hat{K}, \hat{\mathbf{A}})$ into the HJB equation (3.2), we get

$$-\delta + r - r\log(-r\eta\beta) - Q(\hat{p},\hat{K}) = 0\,,$$

yielding the optimal value of $\hat{\beta}$. Moreover, the transversality condition

$$e^{-\delta T}\mathbb{E}[\vartheta(\hat{W}_T)] = e^{-rT}\vartheta(w_0) \to 0, \quad \text{as } T \to \infty$$

holds. Thus, the conditions for the Verification Theorem linking the solution to the HJB equation (3.2) with sufficient conditions for existence of optimal strategies are satisfied, and the desired result follows (see Theorem 9.1 in Section III.9 of Fleming and Soner, 2006). ■

The joint determination of optimal financial asset holdings, $\hat{\mathbf{A}}$, as in the investment literature, and optimal premium rate, $\hat{p}$, as in the premium control literature, represents a unifying framework. As a further generalization, the optimal deductible, $\hat{K}$, is characterized, as well, and the optimal consumption (dividend payout) rate $\hat{c}(w)$ follows. Under further conditions, closed-form solutions are given in Section 4.

By Proposition 3.1, if there exists a pair (p, K) that maximizes the profile Hamiltonian $Q(p, K)$ in (3.13), then the optimal premium, deductible, and investments in risky assets, $(\hat{p}, \hat{K}, \hat{\mathbf{A}})$, are independent of the level of financial wealth, and constant through time. Therefore, and as a result of the CARA assumption, the fraction of wealth invested in risky assets falls as the company accumulates more wealth, with all wealth invested in the money market account in the limit $W_t \to \infty$. Since $Q(p, K)$ does not depend on the company's discount rate, δ, neither do $(\hat{p}, \hat{K}, \hat{\mathbf{A}})$.

Optimal consumption (3.10) is not constant, but fluctuates with income from interest on wealth. Although the level depends on the discount rate, through $\hat{\beta}$ from Proposition 3.1, the marginal propensity to consume depends on neither δ nor wealth, $\hat{c}'(w) = r$. Inserting $\hat{\beta}$ in (3.11), consumption is positive if and only if wealth exceeds the threshold

$$w^\star = \frac{1}{r^2\eta}\left(r - \delta - Q(\hat{p},\hat{K})\right)\,. \tag{3.15}$$

The condition $w^\star \geq 0$ ruling out the event of positive consumption coinciding with negative wealth amounts to

$$\begin{aligned}\delta &\leq r - Q(\hat{p},\hat{K})\\ &= r - \frac{1}{2}\left\|\boldsymbol{\psi}\right\|^2 - r\eta n(\hat{p},\hat{K})\left(b\boldsymbol{\rho}^\top\boldsymbol{\psi} + \hat{p}\right)\\ &\quad + \frac{1}{2}\left[r\eta b n(\hat{p},\hat{K})\right]^2\left(1 - \|\boldsymbol{\rho}\|^2\right) + n(\hat{p},\hat{K})\lambda(\hat{p},\hat{K})q(\hat{K})\,,\end{aligned} \tag{3.16}$$

in which $(\hat{p}, \hat{K})$ do not depend on δ. For δ sufficiently low, the event is eliminated almost surely, because the future matters to the company, and payout can be postponed.

For specific parametrized forms of the functions $n(p, K)$, $\lambda(p, K)$, and $q(K)$, condition (3.16) amounts to parametric restrictions, and we give examples of this in the sequel. Outside these restricted cases, a general upper bound on the probability of positive payout for negative surplus is given by the ruin probability implied by the optimal policy, which reflects the fact that the optimally controlled wealth process becomes negative, $\hat{W}_t < 0$, in finite time (technical ruin) with positive probability (see Browne (1995) for a similar discussion). The ruin probability is analyzed under various assumptions in Appendix A. In the unperturbed case with only the risk-free asset ($I = 0$), $r \geq \delta$ (hence guaranteeing the net profit condition), and exponentially distributed claim size Y, the ruin probability is bounded above by $e^{-r\eta w_0}$, and so becomes arbitrarily small for sufficiently large interest rate, risk aversion, or wealth. In the special case in which, further, the deductible is not controlled by the company, but simply absent (fixed at zero), $r = \delta$, and $r\eta\,\mathbb{E}[Y] < 1 - 1/\sqrt{2}$, the ruin probability is $(1 - r\eta\,\mathbb{E}[Y])e^{-r\eta w_0}$ at the optimal policy in our model.

Remark 3.1. The dynamics of the risky assets (2.3) can be extended without fundamentally changing the nature of the problem. Following Merton (1971), the mean returns $\boldsymbol{\mu}$ and volatility matrix $\boldsymbol{\sigma}$ can be functions of the price processes, in which case $\mathbf{S}_t$ remains a Markov process, and the verification theorem still applies, with the HJB equation modified accordingly. In this case, optimal investments $\hat{\mathbf{A}}$, premium rate $\hat{p}$, and deductible $\hat{K}$ are feedback controls, depending on the prices of the risky assets. Further, following Merton (1976), Liu et al. (2003), Das and Uppal (2004), and Aït-Sahalia et al. (2009), the driving processes can be augmented to include a mixture of Brownian motions and an economy-wide jump process with arrival rate $\bar{\lambda} > 0$ and i.i.d. random jump sizes Z with measure $G(\mathrm{d}z)$. As shown in Appendix B, the optimal premium $\hat{p}$ and deductible $\hat{K}$ in this case can be obtained as the maximizers of the jump-augmented profile Hamiltonian $\tilde{Q}$ in (B.9), which accounts for the expected effects of jumps in asset prices in the company's value function. This can be relevant, e.g., in systemic risk and financial crisis scenarios, with sudden sharp drops in prices (negative jump size). Although investments $\hat{\mathbf{A}}$ are only implicitly determined in the jump-diffusion case, the main results from Proposition 3.1 carry over. Thus, optimal $(\hat{p}, \hat{K}, \hat{\mathbf{A}})$ are independent of the discount rate δ and the level of financial wealth, and thus constant over time. The verification theorem continues to apply, with $\hat{\beta}$ as in Proposition 3.1, and Q replaced by $\tilde{Q}$. For analytical tractability, we focus in Sections 4 and 5 on the pure diffusion case, with constant mean returns and volatility matrix. ■

To be able to access the calculus, we henceforth treat the insurance exposure $n(p, K)$ as continuous and differentiable in (p, K), which should apply to a reasonable degree of approximation for moderate to large market size, N. Accordingly, $n(p, K)/N$ represents the fraction of potential customers purchasing insurance. In this case, the first-order necessary optimality conditions for maximizing $Q(p, K)$ in (3.13) are

$$\frac{\partial n(p,K)}{\partial p}\left[r\eta\Big(p + b\Big[\boldsymbol{\rho}^\top\boldsymbol{\psi} - r\eta b(1 - \|\boldsymbol{\rho}\|^2)n(p,K)\Big]\Big) - \lambda(p,K)q(K)\right] + n(p,K)\left[r\eta - q(K)\frac{\partial\lambda(p,K)}{\partial p}\right] = 0\,, \quad (3.17)$$

$$\frac{\partial n(p,K)}{\partial K}\left[r\eta\Big(p + b\Big[\boldsymbol{\rho}^\top\boldsymbol{\psi} - r\eta b(1 - \|\boldsymbol{\rho}\|^2)n(p,K)\Big]\Big) - \lambda(p,K)q(K)\right] + n(p,K)\left[r\eta\lambda(p,K)\bar{F}_Y(K) + q(K)\left(r\eta\lambda(p,K) - \frac{\partial\lambda(p,K)}{\partial K}\right)\right] = 0\,, \quad (3.18)$$

using $q'(K) = -r\eta(1 - F_Y(K) + q(K))$ for $q(K)$ from (3.7). In general, it is difficult to state conditions under which a critical point $(\hat{p}, \hat{K})$ satisfying the first order conditions (3.17) and (3.18) indeed maximizes $Q(p, K)$. The main difficulty lies in the characterization of the effect of the control variables on the trade-off between portfolio size and profit per customer. Therefore, the next step is to identify appropriate forms of the functions $n(p, K)$, $\lambda(p, K)$, and $q(K)$.

4 Random claim arrival rates

The result from Proposition 3.1 yields optimal investment and consumption, provided an optimal premium and deductible pair maximizing the profile Hamiltonian $Q(p, K)$ in (3.13) can be found. Further analysis of the existence and properties of such a pair $(\hat{p}, \hat{K})$ requires determining suitable forms of the portfolio size $n(p, K)$, the claim arrival rate $\lambda(p, K)$, and the expected net claim valuation function $q(K)$ in (3.7), all of which enter $Q(p, K)$. We derive these three functions from a simple model of the behavior of individual customers, thus ensuring that they are mutually consistent. First, we consider directly the decision problem faced by each potential customer of whether or not to insure at the terms (p, K) offered by the company. Next, we assume that customers are not equally risky. Specifically, while each knows its own claim (casualty) arrival rate, Λ, the insurance company does not have this information. It only knows the distribution across customers, and therefore treats the arrival rate of each potential customer as a random variable. This approach, following ideas from Asmussen et al. (2013) and Thøgersen (2016), delivers the three functions required, thus paving the way for an analysis of optimal $(\hat{p}, \hat{K})$, and by implication optimal investment and consumption, too. The asymmetric information between customer and insurance company implies adverse selection. Because Λ is assumed known to the customer, we say that the customer has complete information. This is relaxed in Section 5.

The individual customer's problem is whether or not to insure at terms (p, K). Without insurance, claims Y_m arriving at rate Λ must be paid by the customer. With insurance, premiums are paid at rate p per period, and claims are only paid up to the deductible, i.e., payments are given by the uninsurable risks $\min\{Y_m, K\}$, again arriving at rate Λ. Thus, the flow of Y_m at rate Λ is compared to the flows of $\min\{Y_m, K\}$ at rate Λ and p. Writing the additional payments that the customer is liable for if uninsured, compared to if insured, as $Y_m - \min\{Y_m, K\} = (Y_m - K)^+$, i.e., the net (of deductible) claims, it is reasonable to compare p to the flow of $(Y_m - K)^+$ arriving at rate Λ. Specifically, when a claim Y_m arrives, the customer always pays the portion $\min\{Y_m, K\}$, whether insured or not, and the issue is whether the customer or the insurance company should cover the remainder, $(Y_m - K)^+$. The customer insures if doing so is preferred over paying the net claims as they arrive, at rate Λ.

Under risk neutrality, the customer insures if this choice is the cheapest, on average, i.e., if $p < p_0(K) := \Lambda \mathbb{E}\left[(Y - K)^+\right]$. Thus, the reservation premium, i.e., the maximum premium that a customer is willing to pay for insurance with a given deductible, is for a risk-neutral customer given by the arrival rate, Λ, times

$$\mathbb{E}\left[(Y - K)^+\right] = \int_K^\infty y F(\mathrm{d}y) - K\bar{F}_Y(K)\,, \tag{4.1}$$

the expected net claim.

Under risk aversion, the reservation premium is naturally higher. In the sequel, we consider specific examples of how to allow for risk aversion on the part of the customer. In these cases, the reservation premium takes the more general form $p_a(K) = \Lambda a(K)$, with $a(K)$ the customer's risk-adjusted expected net claim. Thus, a potential customer buys insurance if and only if

$$p < p_a(K) = \Lambda a(K) . \tag{4.2}$$

This reservation premium policy summarizes customer behavior. Regarding company behavior, we first analyze this for arbitrary $a(K)$, then turn to specific forms of this.

While the insurance company does not know each potential customer's value of Λ, the individual arrival rate of claims, it knows the distribution of this in the population. A standard approach in insurance is to let the company merely provide risk sharing, i.e., as Λ is unknown to the company, the premium charged to customers is given by the net premium, namely, expected frequency times size of net claims, $\mathbb{E}[\Lambda]\mathbb{E}\left[(Y-K)^+\right] = \mathbb{E}[p_0(K)]$, also recognized as the expected risk-neutral reservation premium. A safety loading can be added to this. Allowing for risk averse customers leads to an expected reservation premium $\mathbb{E}[p_a(K)] = \mathbb{E}[\Lambda]a(K)$ instead, based on (4.2). For this to be considered as a candidate premium to be charged to customers, viability of the insurance market requires it exceed the net premium, i.e., $a(K) > \mathbb{E}\left[(Y-K)^+\right]$. It does for a risk averse customer. However, $\mathbb{E}[p_a(K)]$ is only adjusted for customer risk aversion. It ignores both risk aversion on the part of the insurance company and adverse selection, i.e., that the company knows the form of the insurance participation constraint (4.2), and takes into account that only the riskier (higher Λ) potential customers will insure.

For concreteness, let Λ be i.i.d. in the population, with absolutely continuous distribution function $G(z)$, and density function $g(z)$. By (4.2), for given premium rate p and deductible K, the company attracts

$$n(p, K) = N\mathbb{P}\left(\Lambda > \alpha(K)p\right) = N\left[1 - G\left(\alpha(K)p\right)\right] \tag{4.3}$$

customers out of the total population of size N, where $\alpha(K) := 1/a(K)$. The average arrival rate of claims across insured customers is

$$\lambda(p, K) = \mathbb{E}\left[\Lambda \,|\Lambda > \alpha(K)p\right] = \frac{\mathbb{E}\left[\Lambda \mathbb{1}_{\{\Lambda > \alpha(K)p\}}\right]}{1 - G(\alpha(K)p)} = \frac{\int_{\alpha(K)p}^{\infty} z\, G(\mathrm{d}z)}{1 - G(\alpha(K)p)} , \tag{4.4}$$

and the aggregate arrival rate for the collective insurance portfolio is given by the product of (4.3) and (4.4),

$$n(p, K)\lambda(p, K) = N \int_{\alpha(K)p}^{\infty} z g(z)\, dz . \tag{4.5}$$

Henceforth, we assume that the risk-adjusted expected net claim $a(K)$ is differentiable and decreasing, $a'(K) < 0$. This holds trivially under risk neutrality, $a(K) = \mathbb{E}\left[(Y-K)^+\right]$, and is natural under risk aversion, cf. Example 4.1 below. Thus, $\alpha'(K) > 0$, so $\alpha(K)p$ is increasing in both p and K. It follows that while the portfolio size $n(p, K)$ in (4.3) and the aggregate claim rate in (4.5) are decreasing in both p and K, the average claim rate $\mathbb{E}[\Lambda \,|\Lambda > \alpha(K)p]$ in (4.4) is increasing in both. In particular,

$$\frac{\partial\lambda(p,K)}{\partial p} > 0\,, \text{ and } \frac{\partial\lambda(p,K)}{\partial K} > 0\,.$$

This is adverse selection, cf. Rothschild and Stiglitz (1976). Only the riskier customers will continue to purchase insurance if premium or deductible is raised. Thus, the company faces a trade-off in the maximization of (3.13). To illustrate, consider the unperturbed case, $b = 0$, i.e.,

$$Q(p,K) = \frac{1}{2}\,\|\boldsymbol{\psi}\|^2 + r\eta n(p,K)p - n(p,K)\lambda(p,K)q(K)\,. \tag{4.6}$$

An increase in p or K reduces portfolio size $n(p,K)$, and this multiplies on $r\eta p - \lambda(p,K)q(K)$, with $\lambda(p,K)$ increasing in p and K due to adverse selection. Without further assumptions on the distribution of claim sizes, nothing much can be said about the behavior of $q(K)$, and the signs of the derivatives of (4.6) with respect to both p and K remain indeterminate. This leaves scope for finding interior maximizers. Of course, in the perturbed case, $b > 0$, the analysis is further complicated.

For general perturbation of the reserve process, $b \geq 0$, the company's first order conditions for the premium in (3.17) and the deductible in (3.18) reduce to

$$\begin{aligned} r\eta\bar{G}(\alpha(K)p) + \alpha g(\alpha(K)p)\Bigg\{&\alpha(K)q(K)p \\ &+ \Big(1-\|\boldsymbol{\rho}\|^2\Big)\Big(r\eta b\Big)^2 N\bar{G}(\alpha(K)p) - r\eta\Big[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\Big]\Bigg\} = 0\,, \end{aligned}$$

$$\begin{aligned} g(\alpha(K)p)\alpha(K)'p\Bigg\{&\alpha(K)q(K)p + \Big(1-\|\boldsymbol{\rho}\|^2\Big)\Big(r\eta b\Big)^2 N\bar{G}(\alpha(K)p) \\ &- r\eta\Big[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\Big]\Bigg\} + r\eta\lambda(p,K)\bar{G}(\alpha(K)p)\Big[\bar{F}_Y(K) + q(K)\Big] = 0\,, \end{aligned}$$

with $\bar{G} = 1 - G$, and where we have used the partial derivatives

$$\begin{aligned} \frac{\partial n(p,K)}{\partial p} &= -Ng(\alpha(K)p)\alpha(K) < 0\,, \\ \frac{\partial\left(n(p,K)\lambda(p,K)\right)}{\partial p} &= -N\alpha(K)^2 g(\alpha(K)p)p < 0\,, \\ \frac{\partial n(p,K)}{\partial K} &= -Ng(\alpha(K)p)\alpha'(K)p < 0\,, \\ \frac{\partial\left(n(p,K)\lambda(p,K)\right)}{\partial K} &= -N\alpha(K)\alpha'(K)g(\alpha(K)p)p^2 < 0\,. \end{aligned} \tag{4.7}$$

Example 4.1 (Reservation premium using certainty equivalent)**.** Following Asmussen et al. (2013), the reservation premium under risk aversion, $p_a(K) = \Lambda a(K)$ in (4.2), may be determined by assuming that the risk-adjusted expected net claim $a(K)$ is given by the certainty equivalent of the net claim $(Y-K)^+$, as opposed to the expected net claim (4.1). Recall that for a strictly increasing and strictly convex cost function φ, the certainty equivalent Z_φ of a random variable Z satisfies $\varphi(Z_\varphi) = \mathbb{E}[\varphi(Z)]$. By Jensen's inequality, $\mathbb{E}[\varphi(Z)] > \varphi(\mathbb{E}[Z])$, and thus $\varphi(Z_\varphi) > \varphi(\mathbb{E}[Z])$. As φ is increasing, we have $Z_\varphi > \mathbb{E}[Z]$. Thus, with this specification, $a(K) = (Y-K)^+_\varphi > \mathbb{E}\left[(Y-K)^+\right]$, i.e., the risk adjustment increases the reservation premium beyond the net premium. In more detail,

$$a(K) = \varphi^{-1}\left(\mathbb{E}[\varphi((Y-K)^+)]\right) = \varphi^{-1}\left(\int_K^\infty \varphi(y-K)\,F(\mathrm{d}y) + \varphi(0)F_Y(K)\right),$$

and if φ is differentiable, with $\lim_{y\to\infty}\varphi(y)f_Y(y) = 0$, we obtain

$$\varphi'(a(K))a'(K) = \frac{d}{dK}\varphi(a(K)) = -\int_K^\infty \varphi'(y-K)f_Y(y)\,dy\,,$$

using implicit differentiation and Leibniz's rule. As $\varphi'(\cdot) > 0$, this verifies the maintained assumption $a'(K) < 0$ in the certainty equivalence case, and thus $\alpha'(K) > 0$, so the derivatives of portfolio size and aggregate claim rate with respect to premium and deductible are signed as in (4.7). A concrete specification is given by the quadratic cost function $\varphi(z) = \varphi_0 + \varphi_2 z^2$, with $\varphi_2 > 0$. In this case,

$$Z_\varphi = \left(\mathbb{E}\left[Z^2\right]\right)^{1/2} = \left((\mathbb{E}[Z])^2 + \mathrm{Var}[Z]\right)^{1/2} > \mathbb{E}[Z]\,,$$

provided Z is non-degenerate. The risk-adjusted expected net claim is

$$\begin{aligned} a(K) &= \left(\mathbb{E}[(Y-K)^2\mathbb{1}_{\{Y\geq K\}}]\right)^{1/2} \\ &= \left(\int_K^\infty y^2\,F(\mathrm{d}y) - 2K\int_K^\infty y\,F(\mathrm{d}y) + K^2[1-F_Y(K)]\right)^{1/2}, \end{aligned} \tag{4.8}$$

and the customer's reservation premium is $p_a(K) = \Lambda a(K)$, cf. (4.2). Further, using (4.1),

$$\begin{aligned} \alpha'(K) &= a(K)^{-3}\left[\int_K^\infty yF(\mathrm{d}y) - K\bar{F}_Y(K)\right] \\ &= \alpha(K)^3\mathbb{E}\left[(Y-K)^+\right], \end{aligned} \tag{4.9}$$

i.e., proportional to the expected net claim. Asmussen et al. (2013) considered the case $K = 0$, no deductible. ■

Example 4.2 (Reservation premium using principle of equivalent utility)**.** As an alternative to the certainty equivalent method, risk aversion may be introduced into the customer's problem using the principle of equivalent utility. According to this, the customer's reservation premium equates the expected utilities of wealth with and without insurance (see Gerber and Pafum, 1998). In this case, assuming the customer may invest wealth in the risk-free asset at interest rate r, and has preferences over financial wealth of CARA type with constant absolute risk aversion $\zeta > 0$, Thøgersen (2016) approximates the reservation premium as

$$\begin{aligned} p_{\tilde{a}}(K) \approx \Lambda\tilde{a}(K) &= \Lambda\left(\mathbb{E}[(Y-K)^+] + \frac{r\zeta}{2}\mathbb{E}[(Y-K)^2\mathbb{1}_{\{Y\geq K\}}]\right) \\ &= \Lambda\left(\mathbb{E}[(Y-K)^+] + \frac{r\zeta}{2}a(K)^2\right). \end{aligned} \tag{4.10}$$

Thus, the risk-adjusted expected net claim $a(K)$ from the certainty equivalence principle, (4.8), is replaced by $\tilde{a}(K)$. Instead of adjusting the expected net claim up, using the cost function φ as in the certainty equivalence case, a second-moment term is added to it, in analogy with the variance premium principle. In fact, the added term is proportional to the square of the certainty equivalent, $a(K)$, from Example 4.1. ■

In further analysis below, we use the certainty equivalence approach from Example 4.1, along with assumptions on the distributions characterizing the size and frequency of claims, Y and Λ, to derive explicit formulae for portfolio size, claim arrival rate, and expected net claim valuation. A parallel analysis could be carried out based on the principle of equivalent utility from Example 4.2, or other approaches to risk aversion in the customer's problem.

4.1 Exponentially distributed arrival rate

A plausible assumption, e.g., in automobile or fire insurance (cf. Bühlmann and Gisler, 2005, and Denuit et al., 2007) is that Λ is exponentially distributed across customers, with density function $g(z) = \frac{1}{\lambda_0} e^{-\frac{z}{\lambda_0}}$, so that $\mathbb{E}[\Lambda] = \lambda_0$. Henceforth, we adopt this assumption. Thus, (4.3) and (4.4) become

$$n(p, K) = N e^{-\frac{\alpha(K)}{\lambda_0} p}, \text{ and } \lambda(p, K) = \lambda_0 + \alpha(K)p. \tag{4.11}$$

The demand function is the negative exponential, with premium elasticity $\varepsilon(p, K) = -\partial \log n(p, K)/\partial \log p = \alpha(K)p/\lambda_0$, and $\lambda(p, K) = \lambda_0(1 + \varepsilon(p, K))$. Taylor (1986) and Emms et al. (2007) considered the negative exponential demand function, as well, with elasticity proportional to $p/\bar{p}$, interpreting $\bar{p}$ as an average (or market) premium, and not accommodating adverse selection. In contrast, the arrival rate or average riskiness of customers $\lambda(p, K)$ in our analysis increases with the premium in an affine fashion, with slope $\alpha(K)$. Thus, the adverse selection adjustment of the arrival rate, relative to the standard model with constant rate λ_0, is given precisely by the term $\alpha(K)p$ in (4.11), or $\varepsilon(p, K)$ in the equivalent multiplicative representation. Further, (4.11) shows that the demand function and the arrival rate are tied in a subtle manner via $\alpha(K)$ in our analysis. In the demand function, $\bar{p}$ is replaced by $\lambda_0/\alpha(K) = \mathbb{E}[\Lambda]a(K) = \mathbb{E}[p_a(K)]$, the expected reservation premium for a risk averse customer, cf. (4.2). However, this cannot be interpreted as an average market premium. We show next that the optimizing company charges more, due to its own risk aversion and adverse selection.

Let $K^\star := \sup\{K \geq 0 : \alpha(K)q(K) \geq r\eta\}$, with $q(K)$ the expected net claim valuation function from (3.7), and define

$$m(K) := -\frac{\alpha'(K)}{\alpha(K)^2} r\eta + [q(K) + \bar{F}_Y(K)][2r\eta - q(K)\alpha(K)]\,. \tag{4.12}$$

We have the following result.

Theorem 4.1. *Assume that $b = 0$, and that $\alpha(K)$ and $q(K)$ satisfy*

$$q(K)\,,\ \alpha(K)q(K) \to 0 \ \text{ as } K \to \infty\,. \tag{4.13}$$

If there exists a unique positive $\hat{K} \geq K^\star$ such that $m(\hat{K}) = 0$, with $m(K) < 0$ for $K > \hat{K}$, then the pair $(\hat{p}, \hat{K})$ is optimal, with

$$\hat{p} = \frac{r\eta\lambda_0}{\alpha(\hat{K})[r\eta - q(\hat{K})\alpha(\hat{K})]} \,. \tag{4.14}$$

Proof. Under the assumptions on Λ and b, the profile Hamiltonian (4.6) reduces to

$$Q(p, K) = \frac{1}{2}\left\|\boldsymbol{\psi}\right\|^2 + Ne^{-\frac{\alpha(K)p}{\lambda_0}}\Big([r\eta - \alpha(K)q(K)]p - \lambda_0 q(K)\Big) \,. \tag{4.15}$$

For some $K > 0$ sufficiently large, $Q(p, K)$ is strictly positive for all $p \geq 0$, so if $\hat{K}$ maximizes $Q(p, K)$, then $r\eta > \alpha(\hat{K})q(\hat{K})$. Thus, it suffices to maximize $Q(p, K)$ over $p \geq 0$ and $K \geq 0$ satisfying

$$r\eta > \alpha(K)q(K) \,. \tag{4.16}$$

Differentiating with respect to p, we get

$$\frac{\partial Q(p, K)}{\partial p} = Ne^{-\frac{\alpha(K)p}{\lambda_0}}\kappa(p, K) \,,$$

with

$$\kappa(p, K) := r\eta - \frac{\alpha(K)}{\lambda_0}[r\eta - \alpha(K)q(K)]p \,.$$

Therefore, given $K > 0$ satisfying (4.16), the premium rate

$$p(K) = \frac{r\eta\lambda_0}{\alpha(K)[r\eta - q(K)\alpha(K)]}$$

satisfies the first-order condition $\partial Q(p, K)/\partial p = 0$. Since $\partial Q(0, K)/\partial p = Nr\eta > 0$, and $\kappa(\cdot, K)$ decreases strictly to $-\infty$ as $p \to +\infty$, $Q(\cdot, K)$ is strictly increasing on $[0, p(K)]$ and decreasing on $[p(K), +\infty)$. This implies that the critical point $p(K)$ in fact maximizes $Q(\cdot, K)$, so the profile criterion for the remaining variable K is $Q(p(K), K)$. Differentiating this yields

$$\frac{\partial Q(p(K), K)}{\partial K} = N\exp\left(-\frac{r\eta}{r\eta - \alpha(K)q(K)}\right)\frac{r\eta}{r\eta - \alpha(K)q(K)}m(K) \,,$$

i.e., $m(\hat{K}) = 0$ implies that $\hat{K}$ is a critical point, so $m(\cdot)$ is an essential profile gradient, in this sense. From the definition of K^* and the assumptions on $\hat{K}$, it follows that $\partial Q(p(K), K)/\partial K$ is positive on the interval $(K^*, \hat{K})$ and negative for $K > \hat{K}$. Hence, $\hat{K}$ maximizes $Q(p(K), K)$, and the desired result follows from Proposition 3.1. ■

Thus, given an optimal deductible $\hat{K}$, the optimal premium $\hat{p}$ in this case admits the closed form representation (4.14), and the optimal deductible the semi-closed form characterization $m(\hat{K}) = 0$. Optimal consumption (dividend) and investment follow from (3.10) and (3.12), respectively, using Proposition 3.1.

Using that $\alpha(K) = a(K)^{-1}$, it is possible to interpret the optimal premium (4.14) in more detail. Thus, since (4.16) applies at the optimum, we have

$$\hat{p} = \frac{1}{1 - q(\hat{K})/[r\eta a(\hat{K})]}\lambda_0 a(\hat{K}) > \lambda_0 a(\hat{K}) > \lambda_0 \mathbb{E}\left[\left(Y - \hat{K}\right)^+\right] \,. \tag{4.17}$$

The last expression in (4.17) is the net premium, without safety loading, evaluated at the optimal deductible, $\hat{K}$, as selected by the company. It coincides with $\mathbb{E}[p_0(\hat{K})]$, the expected reservation premium for a risk-neutral customer. The last inequality in (4.17) reflects the fact that the expected reservation premium for a risk averse customer, $\mathbb{E}[p_a(\hat{K})] = \lambda_0 a(\hat{K})$, is higher. This is the factor replacing the average or market premium $\bar{p}$ from Taylor (1986) and Emms et al. (2007) in our demand function (4.11). Nevertheless, the optimal premium $\hat{p}$ in our setting is even higher, as reflected in the first inequality in (4.17). The expected reservation premium for a risk averse customer is scaled up further by a factor exceeding unity, by (4.16). This additional adjustment captures the feature that beside the customers, the company (owner-manager) is risk averse, too, and accounts for customer behavior in the optimization. Thus, the scaling factor involves company risk aversion, η, and, in particular, the relative valuation by company and customers of net claims, $q(\hat{K})/a(\hat{K})$. If customers are very risk averse, so that the denominator dominates, and $q(\hat{K})/[r\eta a(\hat{K})]$ is close to zero, then the scaling factor is near unity, and (4.17) shows that the company sets a premium basically determined by the very risk averse customers' expected reservation premium. In contrast, if the company is relatively more risk averse, and hence more concerned about future net claims, reflected in a high value of $q(\hat{K})$ from (3.7), then the scaling factor can be larger. In this case, the company exerts its market power to charge more than customers' expected reservation premium, thus protecting itself from the riskier customers, i.e., the adverse selection aspect.

Both (4.12) and (4.14) involve the functions $\alpha(K)$ and $q(K)$, which in turn depend on the claim size distribution. We consider illustrative examples of this.

Example 4.3 (Gamma distributed claim size)**.** Consider the case that the random claim Y_m follows a gamma distribution with density

$$F(\mathrm{d}y) = \frac{\gamma^\theta}{\Gamma(\theta)} y^{\theta-1} e^{-\gamma y}\,\mathrm{d}y\,, \tag{4.18}$$

with shape parameter $\theta > 0$ and (inverse) scale parameter γ, where $\Gamma(x)$ is the gamma function. Hence, $\mathbb{E}[Y_m] = \theta/\gamma$, and $\mathrm{Var}[Y_m] = \theta/\gamma^2$. We subsequently assume $\gamma > r\eta$. In this case, the truncated moments are

$$\int_K^\infty e^{r\eta y}\,F(\mathrm{d}y) = \left(\frac{\gamma}{\gamma - r\eta}\right)^\theta \frac{\Gamma(\theta,(\gamma-r\eta)K)}{\Gamma(\theta)}\,, \tag{4.19}$$

$$\int_K^\infty y^n\,F(\mathrm{d}y) = \frac{\Gamma(n+\theta,\gamma K)}{\gamma^n\Gamma(\theta)}\,, \tag{4.20}$$

with $\Gamma(s,x)$ the upper incomplete gamma function. Using (4.20) with $n = 1$ and 0 in (4.1) yields the expected net claim

$$\mathbb{E}\left[(Y-K)^+\right] = \frac{\Gamma(1+\theta,\gamma K)}{\gamma\Gamma(\theta)} - K\frac{\Gamma(\theta,\gamma K)}{\Gamma(\theta)}\,.$$

By the recurrence relation $\Gamma(s+1,x) = s\Gamma(s,x) + x^s e^{-x}$, this is compactly recast as

$$\mathbb{E}\left[(Y-K)^+\right] = \frac{1}{\gamma\Gamma(\theta)}\left[\Gamma(\theta,\gamma K)(\theta-\gamma K) + (\gamma K)^\theta e^{-\gamma K}\right]\,. \tag{4.21}$$

Using the certainty equivalent based on the quadratic cost function φ from Example 4.1 for risk adjustment, substitution of (4.20) with $n = 2, 1, 0$ into (4.8) allows calculating the customer's risk-adjusted expected net claim explicitly,

$$a(K) = \left\{ \frac{\Gamma(2+\theta, \gamma K)}{\gamma^2 \Gamma(\theta)} - 2K \frac{\Gamma(1+\theta, \gamma K)}{\gamma \Gamma(\theta)} + K^2 \frac{\Gamma(\theta, \gamma K)}{\Gamma(\theta)} \right\}^{1/2} . \tag{4.22}$$

Thus, the risk adjustment amounts to the difference between (4.22) and (4.21). By one more iteration in the recurrence relation, $\alpha(K) = a(K)^{-1}$ satisfies

$$\alpha(K) = \left\{ \frac{1}{\gamma^2 \Gamma(\theta)} \Big[\Gamma(\theta, \gamma K)[(\theta - \gamma K)^2 + \theta] + e^{-\gamma K} (\gamma K)^\theta [1 + \theta - \gamma K] \Big] \right\}^{-1/2} . \tag{4.23}$$

For the derivative of $\alpha(K)$, combination of (4.9) and (4.21) yields

$$\alpha'(K) = \frac{\alpha(K)^3}{\gamma \Gamma(\theta)} \Big[\Gamma(\theta, \gamma K)(\theta - \gamma K) + (\gamma K)^\theta e^{-\gamma K} \Big] > 0 \,,$$

and $a'(K) < 0$, reconfirming the maintained assumption. Moreover, substituting (4.19) into (3.7), the company's expected net claim valuation is given explicitly as

$$q(K) = e^{-r\eta K} \left(\frac{\gamma}{\gamma - r\eta} \right)^\theta \frac{\Gamma(\theta, (\gamma - r\eta)K)}{\Gamma(\theta)} - \frac{\Gamma(\theta, \gamma K)}{\Gamma(\theta)} . \tag{4.24}$$

With $\alpha(K)$ and $q(K)$ from (4.23) and (4.24) in hand, substitution in (4.12) yields a concrete characterization $m(\hat{K}) = 0$ of the optimal deductible $\hat{K}$, and in (4.14) a closed-form solution for $\hat{p}$, as function of $\hat{K}$. ■

A number of numerical illustrations are provided in this and the following sections, using parameter values chosen with a time unit of one year in mind. Figure 1 sheds light on the behavior of the functions $Q(p, K)$ and $m(K)$ in Theorem 4.1, i.e., the profile Hamiltonian (4.15) and essential gradient (4.12), for gamma distributed claims, as in Example 4.3. Two cases are considered, with common mean claim, $\mathbb{E}[Y] = \theta/\gamma = 1$, but different claim size dispersions. Left exhibits are for $Y \sim \text{Gamma}(0.5, 0.5)$, which implies $\text{Var}[Y] = \theta/\gamma^2 = 2$, and right exhibits for $Y \sim \text{Gamma}(2, 2)$, so $\text{Var}[Y] = 0.5$. In addition to θ and γ, the functions Q and m involve the interest rate, r, set at 5%, and company risk aversion, η, set at 3. Further, Q involves λ_0, set at 0.5, i.e., two years between casualties at mean frequency, and N, normalized at 100, so that $n(p, K)$ represents the percentage of customers insuring. The parameter values correspond to those used in the sensitivity analysis in Section 4.3. For these values, conditions (4.13) and (4.16) are satisfied, with $\alpha(K)$ from (4.23) and $q(K)$ from (4.24). Further, $K^\star = 0.22$ in the low claim size dispersion case, $K^\star = -\infty$ in the high. Although Q in (4.15) involves the market prices of risk, $\boldsymbol{\psi}$, these affect neither the choice of p and K, nor the contours in Figure 1. A dot in the figure indicates the critical point $(\hat{p}, \hat{K}) = (p(\hat{K}), \hat{K})$ maximizing $Q(p, K)$, hence implying $m(\hat{K}) = 0$. The optimal premium rate and deductible are $(\hat{p}, \hat{K}) = (0.51, 2.87)$ in the high claim dispersion case, and $(0.23, 1.69)$ in the low. Thus, greater uncertainty about size of future claims leads the insurance company to increase both premium and deductible.

From the figure, Q is well behaved and, in particular, m has a unique zero. The orientation of the axes of the approximately elliptic contours indicates that p and K

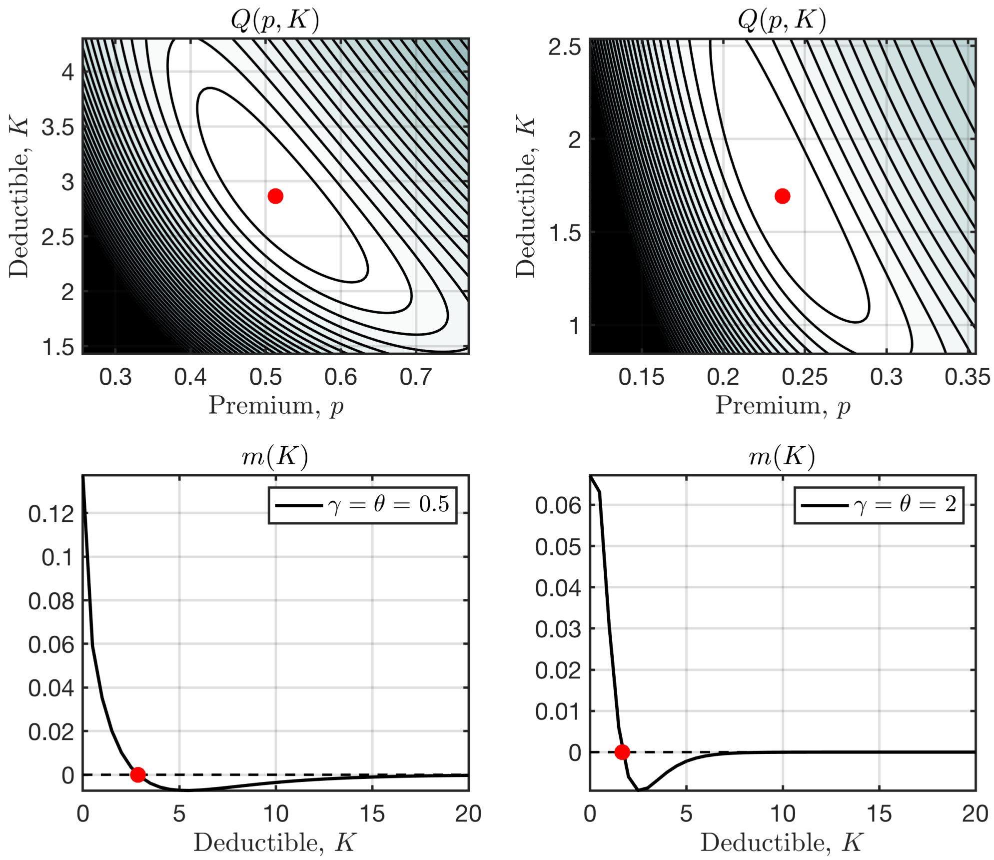

Figure 1. Theorem 4.1 for gamma distributed claims. The figure shows the contours of the profile Hamiltonian $Q(p, K)$ and the essential gradient $m(K)$ for arrival rate of claims $\Lambda \sim \text{Exp}(0.5)$. Left exhibits are for $Y \sim \text{Gamma}(0.5, 0.5)$, and right exhibits for $Y \sim \text{Gamma}(2, 2)$. A dot indicates the critical point $(\hat{p}, \hat{K}) = (p(\hat{K}), \hat{K})$, at which $m(\hat{K}) = 0$. The remaining parameters are $(r, \eta, \lambda_0, N) = (0.05, 3, 0.5, 100)$.

are negatively related around the optimum, i.e., lower deductible implies more insurance and hence higher premium. The risk-adjusted expected net claim $a(\hat{K})$ from (4.22) is 0.74 in the high claim dispersion case, and 0.33 in the low. This is compared to expected net claims of 0.23 and 0.09, based on (4.21), i.e., the risk adjustment amounts to 122% and 267% in the high and low claim dispersion cases, respectively. For comparison, the alternative $\tilde{a}(\cdot)$ specification in (4.10), based on the principle of equivalent utility, combines the already calculated expected net claims with and without risk adjustment, amounting to 0.27 and 0.10 in the high and low dispersion cases, i.e., much less risk adjustment, assuming the same CARA coefficient for customers as for the company, $\zeta = \eta = 3$. The expected reservation premiums for a risk-neutral customer are 0.12 and 0.05 for the two claim dispersion cases, compared to 0.37 and 0.17 if accounting for customer risk aversion, henceforth focusing on the certainty equivalence case, with $a(\hat{K})$ from (4.22). Thus, the latter premiums, which replace $\bar{p}$ from the literature in the demand function, are higher than the premiums under risk neutrality and no safety loading, but lower than the optimal $\hat{p}$, which in addition accounts for risk aversion of the company and adverse selection, by 38% and 35% for the two dispersion levels. Adjustments for both company and customer risk aversion (first and second inequality in (4.17)) constitute large portions of the premium, with the latter adjustment largest in the example.

Company net claim valuations $q(\hat{K})$ are 0.031 and 0.014 in the two dispersion cases, and although clearly measured on a different scale than the customer valuations $a(\hat{K})$ of 0.74 and 0.33 (i.e., utility rather than monetary units), the ratios $q(\hat{K})/a(\hat{K})$ matter for optimal pricing, and (4.16) is satisfied in each case. Substituting $a(\hat{K})$ and $\hat{p}$ in the demand function (4.11) shows that 25% and 24% of potential customers insure in the two cases, i.e., in the example, the company adjusts the contract offered so as to maintain a roughly constant portfolio size, regardless of customer risk. The relatively modest share of potential customers purchasing insurance is indicative of the market power exercised by the optimizing company.

The calculations concern the unperturbed case, $b = 0$, as in Theorem 4.1, i.e., insurance controls $(\hat{p}, \hat{K})$ are chosen separately from risky asset investments. Nevertheless, $(\hat{p}, \hat{K})$ matter for optimal consumption (dividend payout) in (3.10) through the optimized $Q(\hat{p}, \hat{K})$ in $\hat{\beta}$ from Proposition 3.1, and the latter further involves the company's discount rate, δ. To illustrate, set $\delta = 5\%$. Further, $Q(\hat{p}, \hat{K})$ includes $\boldsymbol{\psi}$, and we consider for simplicity $I = 1$ risky asset, with expected return $\mu = 8\%$ and volatility $\sigma = 20\%$. In this case, the market price of risk is 0.15, and optimal investment in the risky asset (3.12) is 5.0. Substitution of $Q(\hat{p}, \hat{K})$ into Proposition 3.1 yields $\hat{\beta} = -1.85 \times 10^{-8}$ and -0.0015 in the high and low dispersion cases. If, for example, initial wealth is $w_0 = 100$, then the optimal consumption path (3.10) starts out at 11.57 in the high and 7.79 in the low claim size dispersion case. The computations illustrate the general point, namely, we provide a unifying framework, with optimal investment, premium, deductible, and payout determined jointly.

Figure 2 illustrates the impact of insurance risk perturbation b, using numerical analysis of $Q(p, K)$ from (3.13) to go beyond Theorem 4.1. Focusing on the high claim size dispersion case, $\mathrm{Var}[Y] = 2$, with the remaining parameters as before (except that δ does not enter Q), the optimal premium and deductible are $(\hat{p}, \hat{K}) = (0.51, 2.87)$ for $b = 0$ (as in the previous figure), $(0.71, 2.56)$ for $b = 0.2$, and $(1.09, 2.22)$ for $b = 0.5$. Here, the results for $b > 0$ require a value for the insurance-finance correlation, and we set $\rho = -0.5$, thus illustrating a systemic risk or contagion case, i.e., insurance risk and financial market losses tend to coincide. The results in the figure show that in case of increased perturbation, i.e., higher b, the company optimally uses the financial markets to hedge the portion of the insurance risk that does not depend on the deductible K, so K can be lowered (note that axes differ), and the premium increased, reflecting both the better insurance product (lower deductible) and the greater uncertainty faced by the company regarding its insurance business. Numerical results (not in the figure) show that this response pattern applies for all non-zero $\rho \in (-1, 1)$. Further, optimal risky investment (3.12) indicates short-selling, at -3.41 for $b = 0.2$ and -5.51 for $b = 0.5$, in spite of the long myopic or speculative position 5.0, shared with the unperturbed case. Thus, hedging demand is negative, at -10.51 and -8.41 in the high and low contagion cases, reflecting the negative insurance-finance correlation. As before, the optimal consumption path could similarly be characterized, for given values of δ and w_0. In effect, the calculations illustrate the general Proposition (3.1), and the interaction

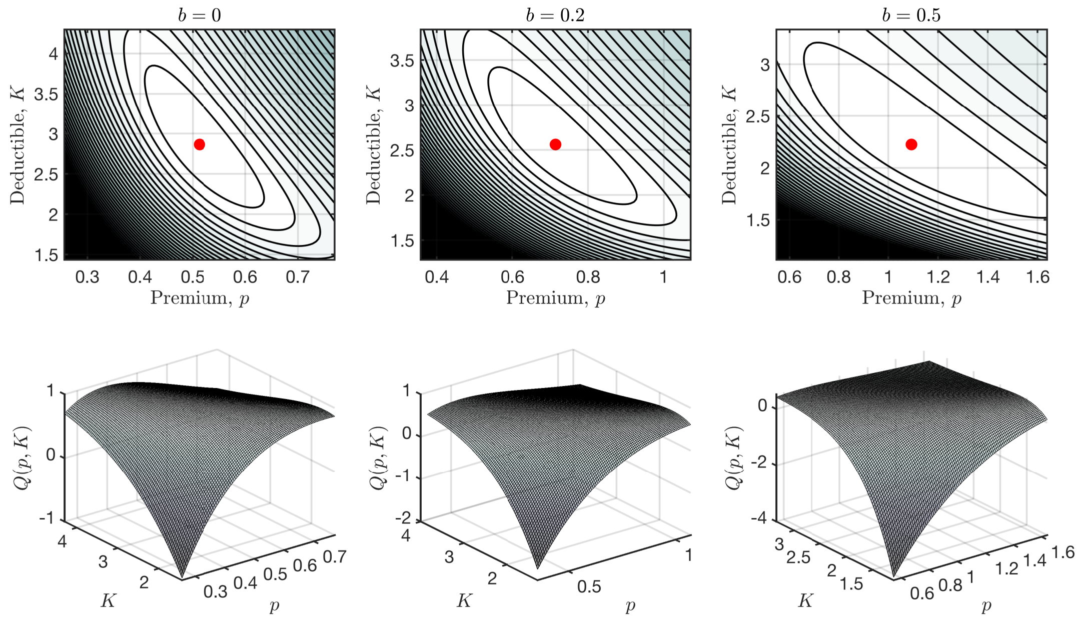

Figure 2. The $Q(p, K)$ function. The figure shows the function $Q(p, K)$ and its contours for different degrees of insurance risk perturbation b. The arrival rate of claims is $\Lambda \sim \mathrm{Exp}(0.5)$, and claim sizes $Y \sim \mathrm{Gamma}(0.5, 0.5)$. A dot indicates the critical point $(\hat{p}, \hat{K})$ maximizing $Q(p, K)$. The remaining parameters are $(r, \eta, I, \mu, \sigma, \rho, \lambda_0, N) = (0.05, 3, 1, 0.08, 0.2, -0.5, 0.5, 100)$.

between investment, premium, and deductible in optimum.

Example 4.4. (Exponentially distributed claims). For $\theta = 1$ in Example 4.3, the truncated moments of the claim size distribution reduce to

$$\int_K^\infty e^{r\eta y}\, F(\mathrm{d}y) = \frac{\gamma e^{-(\gamma - r\eta)K}}{\gamma - r\eta}\,, \tag{4.25}$$

$$\int_K^\infty y^n\, F(\mathrm{d}y) = \frac{n!}{\gamma^n}\, e^{-\gamma K} \sum_{j=0}^{n} \frac{(\gamma K)^j}{j!}\,. \tag{4.26}$$

Substituting (4.26) for $n = 1, 0$ in (4.1) yields the expected net claim $\mathbb{E}\left[(Y-K)^+\right] = e^{-\gamma K}/\gamma$. Using the certainty equivalent based on the quadratic cost function φ from Example 4.1, the customer's risk-adjusted expected net claim $a(K)$, the reciprocal $\alpha(K) = a(K)^{-1}$, and its derivative can be found from

$$\alpha(K) = \frac{\gamma}{\sqrt{2}} e^{\frac{\gamma}{2}K}\,, \text{ and } \alpha'(K) = \frac{\gamma^2}{2\sqrt{2}} e^{\frac{\gamma}{2}K}\,. \tag{4.27}$$

From (4.25), (3.7) and $\bar{F}_Y(K) = e^{-\gamma K}$, the company's expected net claim valuation satisfies

$$q(K) = e^{-r\eta K} \int_K^\infty e^{r\eta y}\, F(\mathrm{d}y) - e^{-\gamma K} = \frac{\frac{r\eta}{\gamma}}{1 - \frac{r\eta}{\gamma}} e^{-\gamma K}\,, \tag{4.28}$$

which together with (4.27) implies that

$$q(K)\alpha(K) = \frac{r\eta e^{-\frac{\gamma}{2}K}}{\sqrt{2}\left(1-\frac{r\eta}{\gamma}\right)} \longrightarrow 0\,, \text{ as } K\to\infty,$$

and hence condition (4.13) holds. Inserting these calculations in (4.12) yields the essential gradient

$$m(K) = r\eta e^{-\frac{3\gamma}{2}K}\left\{-\frac{1}{\sqrt{2}}e^{\gamma K} + \frac{2}{1-\frac{r\eta}{\gamma}}e^{\frac{\gamma}{2}K} - \frac{1}{\sqrt{2}\left(1-\frac{r\eta}{\gamma}\right)^2}\right\}.$$

Using the change of variables $k = e^{\frac{\gamma}{2}K}$, the condition $m(K)=0$ is equivalent to the quadratic equation $B_2k^2 + B_1k + B_0 = 0$, with coefficients

$$B_2 = -\frac{1}{\sqrt{2}}\,, \quad B_1 = \frac{2}{1-\frac{r\eta}{\gamma}}\,, \quad B_0 = -\frac{1}{\sqrt{2}\left(1-\frac{r\eta}{\gamma}\right)^2}\,,$$

and the two real-valued roots

$$k_{(\pm)} = \frac{1}{1-\frac{r\eta}{\gamma}}(\sqrt{2}\pm 1)\,.$$

It follows that $k_{(+)} > 1$, and thus $K_{(+)} = 2\log k_{(+)}/\gamma > 0$. Moreover, in this case we have

$$K^* = \begin{cases} 0\,, & \text{if } \frac{r\eta}{\gamma} < 1-\frac{1}{\sqrt{2}}\,, \\ \frac{2}{\gamma}\log\left[\frac{1}{\sqrt{2}\left(1-\frac{r\eta}{\gamma}\right)}\right]\,, & \text{otherwise,} \end{cases}$$

which is only strictly less than $K_{(+)}$. Since the parabola $B_2k^2+B_1k+B_0$ opens downward, the critical point $K_{(+)}$ is the optimal deductible, as it satisfies the conditions of Theorem 4.1. ∎

For the exponential distribution case from Example 4.4, we have the following sharpening of Theorem 4.1.

Corollary 4.1. *Assume that the arrival rate Λ and size Y_m of claims have exponential distributions with densities $g(z) = \frac{1}{\lambda_0}e^{-\frac{z}{\lambda_0}}$ and $f_Y(y) = \gamma e^{-\gamma y}$, with $\gamma > r\eta$, and that customers' risk-adjusted expected net claim is given by the certainty equivalent $(y-K)^+_\varphi$ based on the quadratic loss function $\varphi(y) = \varphi_0 + \varphi_2 y^2$. Then, the deductible*

$$\hat{K} = \frac{2}{\gamma}\log\left[\frac{1+\sqrt{2}}{1-\frac{r\eta}{\gamma}}\right]$$

and the premium rate

$$\hat{p} = \lambda_0\frac{2\left(1-\frac{r\eta}{\gamma}\right)}{\gamma(1+\sqrt{2})}$$

are optimal.

Proof. Inserting

$$\alpha(K_{(+)}) = \frac{\gamma}{\sqrt{2}} k_{(+)} = \frac{\gamma(1+\sqrt{2})}{\sqrt{2}\left(1 - \frac{r\eta}{\gamma}\right)} \tag{4.29}$$

and

$$r\eta - \alpha(K_{(+)})q(K_{(+)}) = r\eta \frac{1+\sqrt{2}}{2+\sqrt{2}}$$

into (4.14) yields the optimal premium rate. The desired result follows from Theorem 4.1. ■

The corollary illustrates that our approach can deliver closed-form solutions for both optimal premium and optimal deductible, and thus for optimal investments and consumption (dividends), too, using Proposition 3.1. The analytical results make the nature of the optimal policy very transparent. By the corollary,

$$\hat{p} = \frac{\lambda_0}{\gamma/2} e^{-\frac{\gamma}{2}\hat{K}}, \tag{4.30}$$

showing that

$$\frac{\partial \hat{p}}{\partial \hat{K}} = -\lambda_0 e^{-\frac{\gamma}{2}\hat{K}} = -\frac{\gamma}{2}\hat{p} < 0, \tag{4.31}$$

i.e., premium and deductible are inversely related at the optimum, with a higher deductible (less insurance) commanding a lower premium. This analytical result is consistent with the indications from the contour diagrams in Figures 1 and 2 for more general cases.

From (4.29), we have for $\hat{p}$ and $\hat{K}$ from the corollary that

$$\alpha(\hat{K})\hat{p} = \lambda_0\sqrt{2}.$$

Thus, the adverse selection term in (4.11) is proportional to the the population average claim arrival rate, $\mathbb{E}[\Lambda] = \lambda_0$. Therefore,

$$n(\hat{p}, \hat{K}) = Ne^{-\sqrt{2}} \approx N/4, \quad \text{and} \quad \lambda(\hat{p}, \hat{K}) = \lambda_0(1+\sqrt{2}), \tag{4.32}$$

i.e., the optimal size of the insurance portfolio depends only on the number of potential clients, N, and the average arrival rate of claims from a typical client only on λ_0.

The explicit expressions for the optimal policies allow an analytical investigation of parameter dependence. First, while the optimal deductible $\hat{K}$ depends on the company's interest rate r and risk aversion η, as well as the parameter γ governing the claim size distribution, which is natural, it does not depend on the mean claim frequency λ_0. In contrast, the optimal premium $\hat{p}$ depends on all four parameters, $(r, \eta, \lambda_0, \gamma)$. Thus, the company sets the deductible, in essence a truncation of the claim distribution, as a function of the latter, and of company parameters (r, η), but not of the frequency of claims, which is instead compensated for via the optimal premium, in a proportional fashion. Therefore, we have

$$\frac{\partial \hat{K}}{\partial \lambda_0} = 0, \text{ and } \frac{\partial \hat{p}}{\partial \lambda_0} = \frac{\hat{p}}{\lambda_0} > 0, \tag{4.33}$$

i.e., more accidents imply higher premium, not higher deductible.

Secondly, the optimal premium only depends on the company parameters (r,η) through their effect on the optimal deductible, as evident from (4.30). Defining the parameter function $\phi = 2/\left[\gamma(\gamma - r\eta)\right] > 0$, we have

$$\frac{\partial \hat{K}}{\partial r} = \eta\phi > 0\,, \text{ and } \frac{\partial \hat{K}}{\partial \eta} = r\phi > 0\,, \tag{4.34}$$

i.e., the company protects itself through higher deductible in case of higher risk aversion or borrowing cost. Using (4.31), (4.34), and the chain rule, we immediately get

$$\frac{\partial \hat{p}}{\partial r} = -\frac{\gamma}{2}\hat{p}\eta\phi < 0\,, \text{ and } \frac{\partial \hat{K}}{\partial \eta} = -\frac{\gamma}{2}\hat{p}r\phi < 0\,, \tag{4.35}$$

of opposite sign, compared to (4.34), reflecting the negative trade-off between premium and deductible. Further, by direct differentiation of $\hat{K}$ in Corollary 4.1,

$$\frac{\partial \hat{K}}{\partial \gamma} = -\frac{1}{\gamma}(\hat{K} + r\eta\phi) < 0\,, \tag{4.36}$$

i.e., higher γ implies lower mean $1/\gamma$ and variance $1/\gamma^2$ of claim sizes, and thus lower deductible.

As expected, the effect of the claim size distribution on the optimal premium is more complicated, since the reduced exposure (lower mean and dispersion) points to lower premium, and the reduced deductible (4.36) to higher. Specifically, in (4.30), it is necessary to account both for the direct effect of γ on $\hat{p}$, and for the indirect effect of γ via $\hat{K}$. The combined result is

$$\frac{\partial \hat{p}}{\partial \gamma} = \lambda_0 \frac{2\left(1 - 2\frac{r\eta}{\gamma}\right)}{\gamma^2(1+\sqrt{2})}\,, \tag{4.37}$$

and although the maintained assumption $\gamma > r\eta$ guarantees a positive premium $\hat{p}$ in Corollary 4.1, this does not necessarily imply $\gamma > 2r\eta$. Consequently, as the only one of the eight relevant parameter derivatives of $(\hat{p}, \hat{K})$, the sensitivity of the premium with respect to the claim size distribution in (4.37) can take on either sign, depending on parameter values. If $\gamma > 2r\eta$, e.g., for sufficiently low risk aversion η, the company must compensate for the reduced deductible (better insurance offered) in (4.36) through an increased premium, reflecting the negative premium-deductible trade-off in (4.31). However, if $r\eta < \gamma < 2r\eta$, e.g., for high risk aversion, the company values the improvement in claim distribution (reduction in mean and dispersion) so much that it lowers premium and deductible in tandem.

If a company discount rate, $\delta > 0$, as well as parameters of the investment opportunity set beyond r are introduced, then optimal investment and consumption are given in closed form, too, and their dependence on all parameters can be investigated along similar lines. While a detailed analysis is left out here, for space considerations, we briefly outline how to address the issue of positive payout (consumption) for negative surplus (wealth), focusing for simplicity on the case with only risk-free investment $(I = 0)$. Substituting $\hat{K}$ from the corollary in (4.28) yields

$$q(\hat{K}) = \frac{\frac{r\eta}{\gamma}\left(1 - \frac{r\eta}{\gamma}\right)}{(1+\sqrt{2})^2}\,. \tag{4.38}$$

Inserting this, $\hat{p}$ from the corollary, and (4.32) in (3.16), the issue is avoided almost surely if

$$\delta \leq r - \frac{Ne^{-\sqrt{2}}}{1+\sqrt{2}}\lambda_0 \frac{r\eta}{\gamma}\left(1 - \frac{r\eta}{\gamma}\right),$$

which is a simple parametric model restriction. In this case, the company is sufficiently patient. More generally, the probability of positive payout for negative surplus is bounded above by the ruin probability implied by the optimal policy in our model, which in turn is bounded above by $e^{-r\eta w_0}$, cf. Corollary A.1, provided $\delta \leq r$ (hence guaranteeing the net profit condition), and so is arbitrarily small for sufficiently good investment opportunities or high risk aversion, and tends to zero as wealth increases beyond bounds.

4.2 Fixed deductible

Our framework accommodates the special case of a fixed deductible, taken as given and not controlled by the company. The explicit sufficient conditions for the existence of an optimal premium rate are summarized in the following result, for arbitrary degree of insurance risk perturbation, $b \geq 0$.

Theorem 4.2. *Let the deductible be fixed at $K = \overline{K}$. Suppose that $\alpha(\overline{K})$ and $q(\overline{K})$ satisfy $r\eta > \alpha(\overline{K})q(\overline{K})$, and that*

$$1 + \frac{b\alpha(\overline{K})}{\lambda_0}\left[r\eta b N(1 - \|\boldsymbol{\rho}\|^2) - \boldsymbol{\rho}^\top \boldsymbol{\psi}\right] > 0\,. \tag{4.39}$$

Then there exists a unique $\hat{p} > 0$ for which $L(\hat{p}) = 0$, with

$$L(p) := r\eta\left[1 - \frac{b\alpha(\overline{K})}{\lambda_0}\boldsymbol{\rho}^\top\boldsymbol{\psi}\right] - \frac{\alpha(\overline{K})}{\lambda_0}(r\eta - \alpha(\overline{K})q(\overline{K}))p + \frac{\alpha(\overline{K})}{\lambda_0}(r\eta b)^2(1 - \|\boldsymbol{\rho}\|^2)Ne^{-\frac{\alpha(\overline{K})}{\lambda_0}p},$$

and this premium rate $\hat{p}$ is optimal.

Proof. Consider $Q(p, K)$ in (3.13) as a function of p, only, and note that $Q'(p, \overline{K}) = n(p)L(p)$. Condition (4.39) implies that $L(0) > 0$, and $L(p)$ decreases strictly to $-\infty$ as $p \to +\infty$. Thus, there exists a unique $\hat{p}$ such that $L(\hat{p}) = 0$, which is also a critical point of $Q(p, \overline{K})$, and it is in fact optimal, since $L(p)$ (and hence $Q'(p, \overline{K})$) is positive if and only if $p < \hat{p}$. ■

Remark 4.1. In the unperturbed case, $b = 0$, and under the assumptions of Theorem 4.2, the optimal premium rate is given by (4.14) with $\hat{K}$ replaced by $\overline{K}$. In case of gamma distributed claims, as in Example 4.3, this provides a closed-form solution for the optimal premium rate, using (4.23) for $\alpha(\cdot)$ and (4.24) for $q(\cdot)$. This complements the result in Thøgersen (2016) on existence of an optimum premium rate for given fixed deductible under minimization of ruin probability, but with no closed-form solution for the premium. In the special case of exponentially distributed claims ($\theta = 1$), as in Example 4.4, no deductible ($\bar{K} = 0$), $\delta = r$, and $r\eta/\gamma < 1 - 1/\sqrt{2}$, the ruin probability implied by the optimal policy is $(1 - r\eta/\gamma)e^{-r\eta w_0}$, cf. Corollary A.2, thus tightening the upper bound on the probability of positive payout for negative surplus. ■

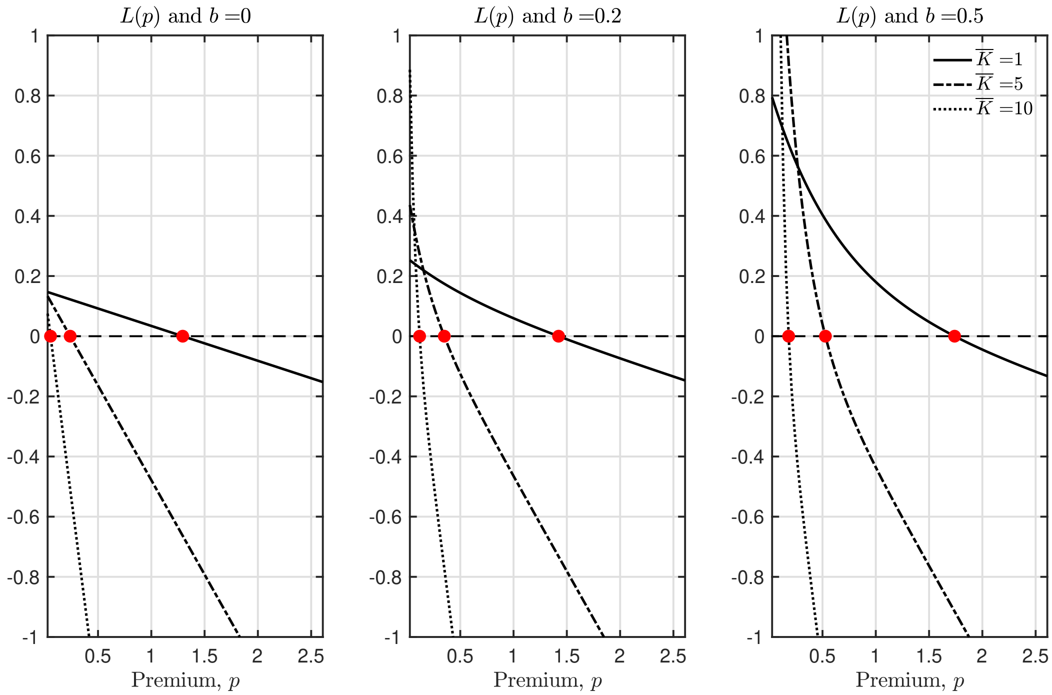

Figure 3. The $L(p)$ function. The figure shows the essential gradient $L(p)$ for different degrees of insurance risk perturbation, b, and different values of the fixed deductible, $\overline{K}$. A dot indicates the optimal premium $\hat{p}$, i.e., the critical point satisfying $L(\hat{p}) = 0$. The arrival rate of claims is $\Lambda \sim \text{Exp}(0.5)$, and claim sizes $Y \sim \text{Gamma}(0.5, 0.5)$. The remaining parameters are $(r, \eta, I, \mu, \sigma, \rho, \lambda_0, N) = (0.05, 3, 1, 0.08, 0.2, -0.5, 0.5, 100)$.

Figure 3 illustrates the properties of the essential gradient $L(p)$ (i.e., $L(p) = 0$ implies $Q'(p, \overline{K}) = 0$) from Theorem 4.2 for different degrees of insurance risk perturbation b and different values of the fixed deductible $\overline{K}$. Claim sizes follow the high dispersion case $Y \sim \text{Gamma}(0.5, 0.5)$ considered in the previous numerical examples, and the values of the remaining parameters follow those examples, too. A dot indicates the optimal premium $\hat{p}$, satisfying $L(\hat{p}) = 0$. From the figure, a lower deductible implies a better insurance contract, hence more customers and higher premium, a demand side effect. Stronger perturbation implies greater uncertainty about the reserve process, and the insurance company protects itself by charging a higher premium, i.e., a supply side effect.

4.3 Sensitivity analysis

Here, we consider the systemic risk or contagion case in more detail. We provide a numerical illustration of the impact of the correlation between the financial and insurance sides of the business, ρ, and the degree of perturbation to the reserve, b, on the optimal premium rate, deductible, asset portfolio allocation, and consumption (dividend) rate. We focus on the high claim size dispersion case from the previous numerical examples, and set the remaining parameters as in those.

Figure 4 shows the results. Naturally, in the unperturbed case, $b = 0$, none of the four controls depends on ρ, as indicated by the solid horizontal line in each exhibit. Next, the negative relation between optimal premium and deductible is confirmed.

For given ρ, stronger insurance risk perturbation, b, implies greater uncertainty about a portion of the reserve that is unrelated to the deductible and can be hedged, so the deductible can be lowered, and the premium increased, reflecting both the lower deductible (more insurance) and greater uncertainty. The magnitude of the impact of such an increase in perturbation on the optimal premium and deductible is reduced as the hedging opportunities offered by the financial markets increase, i.e., as ρ increases in magnitude, hence inducing an (inverted) U-shape of optimal (p) K as function of ρ, for given b. The financial hedge is established through a lower position in the risky asset (selling short, if necessary) for $\rho < 0$, i.e., allocating more of the financial wealth to the money market account in this insurance-finance contagion case, and a higher position for $\rho > 0$ (borrowing in the money market, if necessary). Thus, risky asset investments are monotonic in ρ, and these hedging demands are more pronounced (steeper in ρ) for stronger perturbation, b. At $\rho = 0$, hedging demands vanish, regardless the value of b, and investments reduce to the myopic or speculative demand $\hat{\mathbf{A}} = 5.0$, cf. (3.12), so the curves corresponding to different b intersect at this level for $\rho = 0$.

Finally, as expected, the response of optimal consumption to variation in (ρ, b) is more complicated. The figure shows the initial consumption rate, $c_0 = \hat{c}(w_0)$. For moderate to strong perturbation ($b \geq 0.2$ in the figure), initial consumption is minimal at a certain level of the insurance-finance correlation, $\rho_c(b)$, say, around -0.25 in the figure, corresponding well with the negative systemic risk or contagion scenario for ρ considered in Figures 1 and 2. As ρ increases beyond $\rho_c(b)$, with the company investing larger amounts in the risky asset (reducing the short positions used to hedge insurance risk), consumption increases, because $\mu > r$, i.e., the equity premium is positive, and the investment strategy now earns a higher expected return (cf. Merton, 1971). For given ρ, consumption is lower for stronger perturbation b, at least for b above a certain level, $b_c(\rho)$, say (e.g., for $b \geq 0.05$ at $\rho = 0.5$ in the figure). However, the increase in the consumption (dividend) rate with the magnitude of ρ (more precisely, with the magnitude of the deviation in ρ from $\rho_c(b)$) is greater if hedging opportunities are more important, as captured by the perturbation or contagion parameter b.

5 Customers with partial information

The results in the previous section assume that the customers have complete information in the sense that they know with certainty their frequency of casualty occurrences. In practice, however, customers can be overly optimistic or pessimistic regarding own risk. Following Asmussen et al. (2013), we allow for this possibility by assuming that the customer's assessment of own arrival rate takes the form ΛS, with S a random variable independent of Λ, representing the optimism/pessimism of the customer. The decision by a potential customer of whether or not to purchase insurance is now based upon ΛS rather than the true Λ, so customers have partial information, only. Assuming the distribution of S is i.i.d. across customers and inverse gamma, $1/S$ has a gamma distribution, with density $f_{1/S}(x) = (\zeta^\tau / \Gamma(\tau)) x^{\tau-1} e^{-\zeta x}$, with shape parameter τ and (inverse) scale parameter ζ. Finite mean, $\mathbb{E}[S] < \infty$, requires $\tau > 1$, and finite variance, $\mathrm{Var}[S] < \infty$, requires $\tau > 2$. We assume $\tau > 1$, i.e., customers can be considered optimistic or pessimistic on average according to whether $\mathbb{E}[S]$ is below or above unity. Then, for a given premium rate, p, and

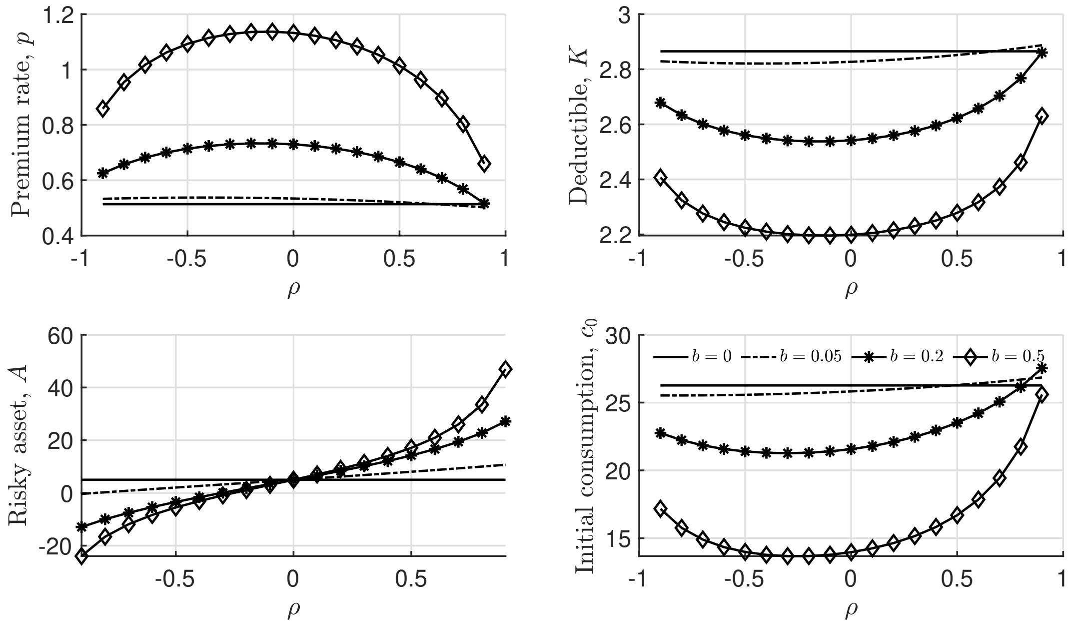

Figure 4. Optimal policies. The figure shows the optimal premium rate, p, deductible, K, risky asset investment, A, and initial consumption, c_0, for different values of the insurance-finance correlation, ρ, and degree of insurance risk perturbation, b. The arrival rate of claims is $\Lambda \sim \mathrm{Exp}(0.5)$, and claim sizes $Y \sim \mathrm{Gamma}(0.5, 0.5)$. The remaining parameters are $(r, \delta, \eta, I, \mu, \sigma, \lambda_0, N) = (0.05, 0.05, 3, 1, 0.08, 0.2, 0.5, 100)$.

deductible K, the company attracts

$$n(p, K) = N\mathbb{P}(\Lambda S > \alpha(K)p) = N\mathbb{P}\left(\Lambda > \frac{\alpha(K)p}{S}\right) = N\left(\frac{\lambda_0\zeta}{\lambda_0\zeta + \alpha(K)p}\right)^{\tau} \tag{5.1}$$

customers, using that $\mathbb{P}\left(\Lambda > \frac{x}{S}\right) = \mathbb{E}\left[e^{-\frac{x}{\lambda_0 S}}\right]$. Thus, the demand function is now generalized hyperbolic, as opposed to the negative exponential demand function in (4.11). Under partial information, the average arrival rate of claims is given by

$$\lambda(p, K) = \mathbb{E}[\Lambda \,|\, \Lambda S > \alpha(K)p] = \lambda_0 + \frac{\lambda_0\alpha(K)\tau p}{\lambda_0\zeta + \alpha(K)p}\,, \tag{5.2}$$

in which the last term, excess expected arrivals beyond λ_0, is a combined adverse selection ($\alpha(K)p$) and partial information (ζ, τ) effect (see Appendix C for the derivation of (5.1)-(5.2)). Under the maintained assumption that the customer's risk-adjusted expected net claim $a(K)$ is differentiable with $a'(K) < 0$, it follows that

$$\frac{\partial n(p,K)}{\partial p} = -\frac{\tau\alpha(K)n(p,K)}{\lambda_0\zeta + \alpha(K)p} < 0\,, \quad \frac{\partial n(p,K)}{\partial K} = -\frac{\tau\alpha'(K)n(p,K)p}{\lambda_0\zeta + \alpha(K)p} < 0\,,$$
$$\frac{\partial \lambda(p,K)}{\partial p} = \frac{\lambda_0^2\alpha(K)\tau\zeta}{(\lambda_0\zeta + \alpha(K)p)^2} > 0\,, \quad \frac{\partial \lambda(p,K)}{\partial K} = \frac{\lambda_0^2\alpha'(K)\tau\zeta p}{(\lambda_0\zeta + \alpha(K)p)^2} > 0\,.$$

Similarly to the complete information case, Section 4, the number of customers, $n(p, K)$, drops with increases in the premium rate and the deductible, while the average frequency of claims, $\lambda(p, K)$, increases due to adverse selection. Using these results, it is straightforward to show that the expected arrival rate of claims from the aggregate insurance portfolio, $n(p,K)\lambda(p,K)$, declines as the premium rate or deductible is raised, as in (4.7).

Using these partial derivatives and factoring out $n(p,K)$, the company's first order conditions (3.17) and (3.18) reduce to

$$\begin{aligned}
r\eta\Big(\lambda_0\zeta + \alpha(K)p\Big)^2 - r\eta\tau\alpha(K)\Big(\lambda_0\zeta + \alpha(K)p\Big)\Big[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\Big] \\
+ \Big(1 - \|\boldsymbol{\rho}\|^2\Big)\Big(r\eta b\Big)^2\tau\alpha(K)n(p,K)\Big(\lambda_0\zeta + \alpha(K)p\Big) \\
+ q(K)\lambda_0\alpha(K)^2\tau(\tau+1)p = 0\,, \quad (5.3)
\end{aligned}$$

$$\begin{aligned}
- r\eta\tau\alpha'(K)p\Big(\lambda_0\zeta + \alpha(K)p\Big)\Big[b\boldsymbol{\rho}^\top\boldsymbol{\psi} + p\Big] \\
+ \Big(1 - \|\boldsymbol{\rho}\|^2\Big)\Big(r\eta b\Big)^2\tau\alpha'(K)n(p,K)p\Big(\lambda_0\zeta + \alpha(K)p\Big) \\
+ r\eta\lambda_0\Big(\bar{F}_Y(K) + q(K)\Big)\Big(\lambda_0\zeta + \alpha(K)p\Big)\Big[\lambda_0\zeta + \alpha(K)(1+\tau)p\Big] \\
+ \alpha(K)\alpha'(K)q(K)\lambda_0(1+\tau)\tau p^2 = 0\,. \quad (5.4)
\end{aligned}$$

Figure 5 illustrates the properties of the profile Hamiltonian $Q(p,K)$ from (3.13), now using (5.1) for portfolio size $n(p,K)$ and (5.2) for average arrival rate $\lambda(p,K)$. The figure further illustrates the associated first order conditions (5.3) and (5.4). Customers' beliefs about own claim arrival rates are characterized by $S \sim \text{Inv-Gamma}(4,4)$, implying that $\mathbb{E}[S] = 1.33$, i.e., a case of pessimistic customers, on average, and $\text{Var}[S] = 0.88$. Adopting the high claim size dispersion case from the previous figures, $Y \sim \text{Gamma}(0.5, 0.5)$, $\alpha(K)$ in (5.1)-(5.2) follows from (4.23), and $q(K)$ in (3.13) from (4.24), as in the complete information case. The values of the remaining parameters can be found in the caption below the figure. A dot indicates the optimal premium $\hat{p}$ and deductible $\hat{K}$ maximizing $Q(p,K)$. The optimal values are $(\hat{p},\hat{K}) = (1.12, 1.70)$. Again, optimal asset holdings and consumption follow from Proposition 3.1. Compared to the left exhibits of Figure 1, i.e., the corresponding complete information case, where $(\hat{p},\hat{K}) = (0.51, 2.87)$, pessimistic customers in the partial information extension on average purchase more insurance, thus leading the optimizing company to increase the premium, and at the same time reduce the deductible, consistent with the overall negative premium-deductible trade-off.

The analysis illustrates the flexibility and generality of the approach to accommodate alternative portfolio size and claim frequency specifications, and still obtain jointly optimal premium, deductible, investment, and consumption (dividend) controls.

5.1 Fixed deductible

Consider again the special case in which the deductible is not controlled by the company, but instead fixed at $K = \overline{K}$, cf. Section 4.2. In this case, the first order condition (5.3) for the optimal premium can be compactly written as

$$M(p,\overline{K}) := M_2(\overline{K})p^2 + M_1(\overline{K})p + M_0(\overline{K}) + \frac{(r\eta b)^2(1-\|\boldsymbol{\rho}\|^2)(\lambda_0\zeta)^\tau N\tau\alpha(\overline{K})}{(\lambda_0\zeta + \alpha(\overline{K})p)^{\tau-1}} = 0\,, \quad (5.5)$$

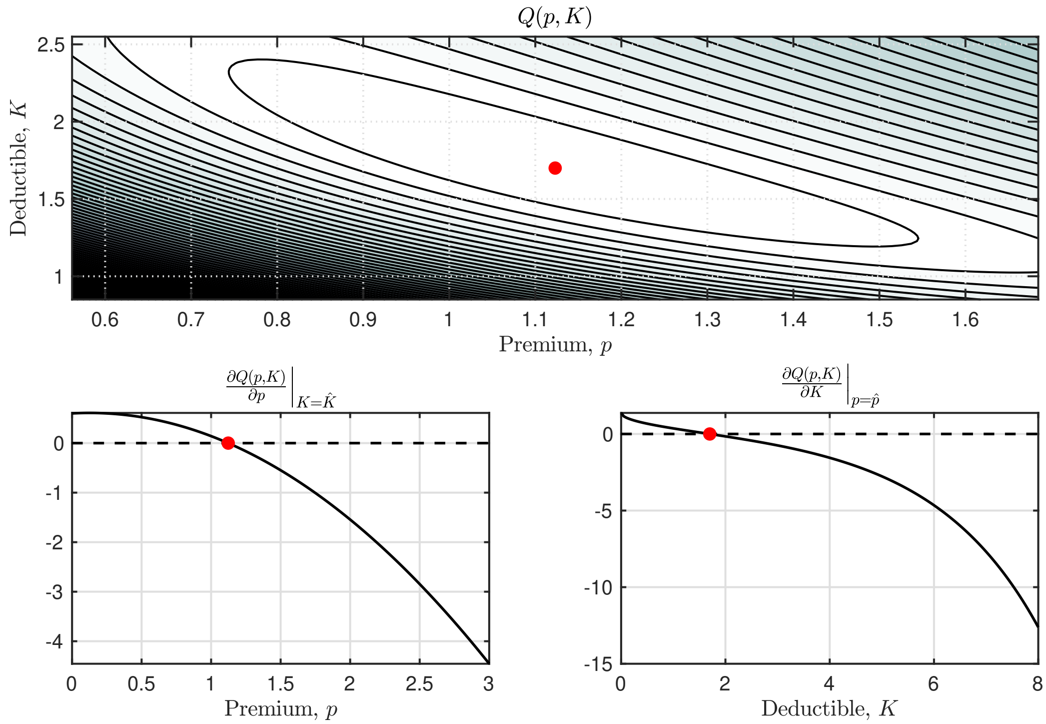

Figure 5. The $Q(p,K)$ function under partial information. The top exhibit shows contours of the profile Hamiltonian $Q(p,K)$. The bottom exhibits illustrate the first order conditions with respect to the premium rate p and deductible K. Dots indicate the critical points $\hat{p}$ and $\hat{K}$ for which $\partial Q(\hat{p},K)/\partial p = \partial Q(p,\hat{K})/\partial K = 0$. The arrival rate of claims is $\Lambda \sim \text{Exp}(0.5)$, the customer's partial information variable $S \sim \text{Inverse-Gamma}(4,4)$, and claims sizes $Y \sim \text{Gamma}(0.5, 0.5)$. The remaining parameters are $(r, \eta, b, \lambda_0, N) = (0.05, 3, 0, 0.5, 100)$.

with

$$
\begin{aligned}
M_2(\overline{K}) &:= r\eta\alpha(\overline{K})^2(1-\tau)\,,\\
M_1(\overline{K}) &:= \lambda_0\alpha(\overline{K})[\alpha(\overline{K})q(\overline{K})\tau(1+\tau) + r\eta\zeta(2-\tau)] - \alpha(\overline{K})^2\tau r\eta b\boldsymbol{\rho}^\top\boldsymbol{\psi}\,,\\
M_0(\overline{K}) &:= r\eta\lambda_0\zeta[\lambda_0\zeta - \alpha(\overline{K})\tau b\boldsymbol{\rho}^\top\boldsymbol{\psi}]\,.
\end{aligned}
$$

We have the following result, for arbitrary perturbation to the reserve process, $b \geq 0$.

Theorem 5.1. *Suppose that*

$$
\lambda_0\zeta + r\eta b^2(1-\|\boldsymbol{\rho}\|^2)N\tau\alpha(K) > \alpha(K)\tau b\boldsymbol{\rho}^\top\boldsymbol{\psi}\,, \tag{5.6}
$$

and that there exists a unique $\hat{p} > 0$ such that $M(\hat{p}, \overline{K}) = 0$. Then $\hat{p}$ is optimal.

Proof. Consider $Q(p,\overline{K})$ in (3.13) and $M(p,\overline{K})$ in (5.5) as functions of p, only. If $M(\hat{p},\overline{K}) = 0$, then $\hat{p}$ is a critical point of $Q(p,\overline{K})$. Moreover, $M(0,\overline{K}) > 0$ under (5.6). Then $M(p,\overline{K})$, and hence $Q'(p,\overline{K})$, is positive only for $p \leq \hat{p}$, which implies that $\hat{p}$ is optimal. ■

Remark 5.1. If $\hat{p} > -\frac{1}{2}\left(\frac{M_1(\overline{K})}{M_2(\overline{K})}\right)$, then $M'(p,\overline{K})$ is negative for $p > \hat{p}$, and so is $M(p,\overline{K})$. ■

Since $M_0(\overline{K})$ and $M_1(\overline{K})$ involve the market price of risk $\boldsymbol{\psi}$ for $b > 0$, so does the optimal premium $\hat{p}$ in Theorem 5.1. Again, optimal investment and consumption follow from Proposition 3.1. Finally, for $b = 0$, the optimal premium admits a closed-form representation, as stated in the following result.

Corollary 5.1. *Assume $b = 0$. Then $M(p, \overline{K})$ in (5.5) is a quadratic equation in p. Under the assumptions of Theorem 5.1, the optimal premium rate is*

$$\hat{p}(\overline{K}) = \frac{\lambda_0}{2r\eta\alpha(\overline{K})(1-\tau)} \left[\widetilde{M}_1(\overline{K}) + \sqrt{\widetilde{M}_1(\overline{K})^2 + 4(r\eta\zeta)^2(\tau - 1)} \right],$$

with $\widetilde{M}_1(\overline{K}) = \alpha(\overline{K})q(\overline{K})\tau(1+\tau) + r\eta\zeta(2-\tau)$.

For gamma distributed claims, $\alpha(K)$ and $q(K)$ are given by (4.23) and (4.24), so in this case, for given fixed deductible $\overline{K}$, the corollary provides the closed-form solution for the optimal premium, $\hat{p}$, and hence for optimal investment and consumption (dividend), too. For general claim size distributions, $\alpha(K) = a(K)^{-1}$ and $q(K)$ in the corollary are determined from (4.8) and (3.7).

6 Conclusion

Systemic risk and contagion between financial and insurance markets is a very realistic and relevant phenomenon, and should be accounted for in the management of insurance companies. Our analysis provides a unifying framework for financial investment and premium control, with simultaneous optimal choice of the deductible, as well as optimization of the path for consumption, or dividend payout to the owner-manager, assuming the latter exhibits constant absolute risk aversion. Optimal risky asset holdings reflect both speculative financial investments and a hedge against insurance risk. Insurance demand reflects risk aversion on the customer side, too, as well as adverse selection, i.e., an increase in either premium or deductible reduces portfolio size and increases the riskiness of the average customer. Premium rate and deductible are inversely related around the optimum. The optimal premium exceeds the net premium, namely, expected frequency time size of net (of deductible) claims, which would correspond to standard insurance pricing, i.e., risk sharing, but without safety loading, and also to the expected reservation premium of hypothetical risk-neutral customers. The optimal premium further exceeds the higher expected reservation premium of customers after appropriately accounting for their risk aversion, although this is the level replacing the average or market premium from the literature in the demand function. The extent of the excess of the optimal premium beyond customers' expected risk-adjusted reservation premium reflects the company's risk aversion, the relative valuation of net claims by company and customers, and the market power exerted by the company to protect itself against risky customers, i.e., the adverse selection aspect. In the complete information case, customers know their own riskiness (casualty arrival rate), but the company only knows the distribution of rates across customers, i.e., there is asymmetric information. In the partial information extension, customers can be overly optimistic or pessimistic about own riskiness. Closed-form solutions for optimal investment, premium, deductible, and payout are obtained under specific assumptions on the distributions of size and frequency of claims.

Overall, the results should encourage insurance companies to manage risks stemming from the financial and insurance sides of their businesses in an active manner, and jointly. Future research could investigate (i) alternative utility functions, e.g., constant relative risk aversion; (ii) separation of ownership and control of the insurance company; (iii) accounting explicitly for the possibility of ruin in the company's optimization problem, possibly using minimization of ruin probability as an alternative criterion (see Appendix A); (iv) different types of deductibles, e.g., the proportional deductible; (v) using the principle of equivalent utility rather than the certainty equivalent for the reservation premium, cf. Examples 4.1 and 4.2; and (vi) competition from other insurance companies. Although beyond the scope of the present paper, all these variations offer exciting avenues for further investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Bent Jesper Christensen and Juan Carlos Parra-Alvarez are also research fellows at the Danish Finance Institute (DFI), the Center for Research in Econometric Analysis of TimE Series (CREATES), and the Dale T. Mortensen Center (DTMC). Part of this research was done while Rafael Serrano was visiting Aarhus University, and he gratefully acknowledges financial support from DTMC. We are also grateful to the anonymous reviewers for helpful comments.

Appendix

A Ruin probability

Consider the optimal policy in our model, i.e., $(\hat{c}, \hat{p}, \hat{K}, \hat{\mathbf{A}})$ from Proposition 3.1. Let

$$\hat{\tau}(w_0) \equiv \tau_{\hat{c},\hat{\mathbf{A}},\hat{p},\hat{K}}(w_0) := \inf\left\{t > 0 : \hat{W}_t < 0 | \hat{W}_0 = w_0\right\}$$

denote the time of ruin, given this policy, and

$$\hat{\Psi}(w_0) \equiv \Psi(w_0|\hat{c}, \hat{\mathbf{A}}, \hat{p}, \hat{K}) := \mathbb{P}(\hat{\tau}(w_0) < \infty)$$

the probability of ultimate ruin in finite time. Here, $\hat{W}_t$ is the optimally controlled wealth process, obtained by substituting $\hat{\beta}$ from Proposition 3.1 and $\hat{\mathbf{A}}$ from (3.12) in (3.14), to yield the perturbed Cramér-Lundberg process

$$\begin{aligned}\mathrm{d}\hat{W}_t = \frac{1}{r\eta}\left(r - \delta + \frac{1}{2}\|\boldsymbol{\psi}\|^2 + \frac{1}{2}(r\eta b\hat{n})^2\left(1 - \|\boldsymbol{\rho}\|^2\right) + \hat{n}\hat{\lambda}\hat{q}\right)\mathrm{d}t \\ + \left(\frac{1}{r\eta}\boldsymbol{\psi}^\top + \hat{n}b\boldsymbol{\rho}^\top\right)\mathrm{d}\mathbf{B}_t - \hat{n}b\mathrm{d}\overline{B}_t - \int_{\mathbb{R}_+}(y - \hat{K})^+ N^{\hat{p},\hat{K}}(\mathrm{d}y, \mathrm{d}t)\,,\end{aligned}$$

with aggregate jump intensity $\hat{n}\hat{\lambda}$, writing $\hat{n} = n(\hat{p}, \hat{K})$, $\hat{\lambda} = \lambda(\hat{p}, \hat{K})$, and $\hat{q} = q(\hat{K})$. It is well known that $\hat{\Psi}(w_0) < 1$ for all $w_0 \geq 0$ under the so-called *net profit condition*, which in the present case amounts to

$$\frac{1}{r\eta}\left\{r - \delta + \frac{1}{2}\|\boldsymbol{\psi}\|^2 + \frac{1}{2}(r\eta b\hat{n})^2\left(1 - \|\boldsymbol{\rho}\|^2\right) + \hat{q}\hat{n}\hat{\lambda}\right\} > \hat{n}\hat{\lambda}\mathbb{E}[(Y - K)^+]\,. \tag{A.1}$$

If $\delta \leq r$, this holds automatically, since $q(K) \geq r\eta\mathbb{E}[(Y - K)^+]$, so the ruin probability is below one at the expected discounted utility maximizing strategy in this case.

In the general light-tailed case where claim sizes have exponential moments, the so-called "small claim case", the ruin probability decays exponentially fast in the initial reserve (see Asmussen and Albrecher, 2010). Indeed, let $m_Y(z) := \mathbb{E}[e^{z(Y-\hat{K})^+}]$ denote the moment-generating function (m.g.f.) of claim size net of optimal deductible, and define

$$\begin{aligned}\Theta(z) := \hat{n}\hat{\lambda}\left[m_Y(z) - 1\right] - \frac{1}{r\eta}\left\{r - \delta + \frac{1}{2}\|\boldsymbol{\psi}\|^2 + \frac{1}{2}(r\eta b\hat{n})^2\left(1 - \|\boldsymbol{\rho}\|^2\right) + \hat{q}\hat{n}\hat{\lambda}\right\}z \\ + \frac{1}{2}\left\{\frac{1}{(r\eta)^2}\|\boldsymbol{\psi}\|^2 + (\hat{n}b)^2\left(1 - \|\boldsymbol{\rho}\|^2\right)\right\}z^2\,.\end{aligned} \tag{A.2}$$

If there exists $z^* > 0$ such that $\Theta(z^*) = 0$, then Lundberg's inequality

$$\hat{\Psi}(w_0) < e^{-z^* w_0} \tag{A.3}$$

holds, where z^* is called the Cramér-Lundberg adjustment coefficient (see Xu et al. (2018) for a proof in a related setting). In this case, given $\varepsilon \in (0, 1)$, we have $\hat{\Psi}(w_0) \leq \varepsilon$ if

$$z^* w_0 \geq -\log\varepsilon\,,$$

which holds for sufficiently large initial wealth, $w_0 \geq \frac{1}{z^*} \log \frac{1}{\varepsilon}$, or if, given $w_0 > 0$, model parameters are calibrated to satisfy

$$z^* \geq \frac{1}{w_0} \log \frac{1}{\varepsilon} .$$

In the following, we focus on the case in which the risk-adjusted expected net claim is determined using the certainty equivalent based on a quadratic cost function, as in Example 4.1,

$$a(K) = \left(\mathbb{E}[(Y-K)^2 \mathbf{1}_{\{Y \geq K\}}]\right)^{1/2} ,$$

and the arrival rate of claims, Λ, is exponentially distributed with $\mathbb{E}[\Lambda] = \lambda_0$, cf. Section 4.1. In this case,

$$n(p, K) = N \exp\left(-\Phi\right) , \quad \lambda(p, K) = \lambda_0 \left(1 + \Phi\right) ,$$

with $\Phi := p/\lambda_0 a(K)$ the premium elasticity.

A.1 Exponentially distributed claims

Consider the setting of Example 4.4, i.e., $Y \sim \mathbb{E}(\gamma)$, with density (4.18) for $\theta = 1$, and $b = 0$. Using the closed-form expression for the optimal pair $(\hat{p}, \hat{K})$ in Corollary 4.1, we have from (4.29) that

$$\alpha(\hat{K}) := a(\hat{K})^{-1} = \frac{\gamma}{\sqrt{2}} \frac{1+\sqrt{2}}{1-\frac{r\eta}{\gamma}} ,$$

and from (4.32) and (4.38) that

$$\hat{n} = N e^{-\sqrt{2}}, \quad \hat{\lambda} = \lambda_0 (1+\sqrt{2}), \quad \hat{q} = \frac{\frac{r\eta}{\gamma}\left(1-\frac{r\eta}{\gamma}\right)}{(1+\sqrt{2})^2} .$$

Further, the m.g.f of the claim sizes net of deductible is given by

$$m_Y(z) = \frac{z}{\gamma - z} e^{-\gamma \hat{K}} + 1 .$$

Using these and $\mathbb{E}[(Y-\hat{K})^+] = e^{-\gamma\hat{K}}/\gamma = \left(1-\frac{r\eta}{\gamma}\right)^2 / \gamma(1+\sqrt{2})^2$ in (A.2) yields

$$\Theta(z) := \hat{n}\hat{\lambda}\mathbb{E}[(Y-\hat{K})^+] \frac{\gamma z}{\gamma - z} - \frac{1}{r\eta}\left(r - \delta + \frac{1}{2}\left\|\boldsymbol{\psi}\right\|^2 + \hat{q}\hat{n}\hat{\lambda}\right) z + \frac{1}{2(r\eta)^2}\left\|\boldsymbol{\psi}\right\|^2 z^2 ,$$

with $\Theta(0) = 0$, $\Theta'(0) = \hat{n}\hat{\lambda}\mathbb{E}[(Y-\hat{K})^+] - \frac{1}{r\eta}\left(r - \delta + \frac{1}{2}\left\|\boldsymbol{\psi}\right\|^2 + \hat{q}\hat{n}\hat{\lambda}\right) < 0$, and $\Theta''(0) = \left(\hat{n}\hat{\lambda}/\gamma\right)\mathbb{E}[(Y-\hat{K})^+] + \left\|\boldsymbol{\psi}\right\|^2/(r\eta)^2 > 0$. Thus, there exists $z^\star > 0$ that solves the quadratic equation $\Theta(z^*) = 0$.

For illustration, consider the case in which the insurance company only reinvests any surplus at the risk-free rate. This is equivalent to assuming $\hat{\mathbf{A}} = \mathbf{0}$. The condition $\Theta(z) = 0$ then reduces to the linear equation

$$\hat{n}\hat{\lambda}\mathbb{E}[(Y-\hat{K})^+] \frac{\gamma z}{\gamma - z} - \frac{1}{r\eta}\left(r - \delta + \hat{q}\hat{n}\hat{\lambda}\right) z = 0 ,$$

with closed form solution

$$z^\star = \gamma\left(1 - \frac{Ne^{-\sqrt{2}}\lambda_0 \frac{r\eta}{\gamma}\left(1-\frac{r\eta}{\gamma}\right)^2}{(r-\delta)(1+\sqrt{2}) + Ne^{-\sqrt{2}}\lambda_0\frac{r\eta}{\gamma}\left(1-\frac{r\eta}{\gamma}\right)}\right)$$

under the net profit condition (A.1). The latter in the present case reads

$$r - \delta + \frac{Ne^{-\sqrt{2}}\lambda_0\left(\frac{r\eta}{\gamma}\right)^2\left(1-\frac{r\eta}{\gamma}\right)}{1+\sqrt{2}} > 0\,.$$

Again, this is automatic under the relative patience assumption $\delta \le r$, and we have the following result.

Corollary A.1. *Under the assumptions of Corollary 4.1, if $\delta \le r$, then $z^\star \ge r\eta$, and the probability of ultimate ruin is bounded above by*

$$\hat{\Psi}(w_0) \le e^{-r\eta w_0}\,.$$

A.2 No deductible

As in Section 4.2, consider the case that the company is not controlling the deductible, and assume this is in fact absent, $\overline{K} = 0$. Further, relax the assumption on the claim size distribution to $Y \sim \text{Gamma}(\theta, \gamma)$, as in Example 4.3. The m.g.f of claim sizes is

$$m_Y(z) = \left(\frac{\gamma}{\gamma - z}\right)^\theta\,, \quad z < \gamma\,.$$

Under the conditions

$$r\eta > \frac{m_Y(r\eta) - 1}{(\mathbb{E}[Y^2])^{1/2}}$$

and

$$1 + \frac{b}{\lambda_0(\mathbb{E}[Y^2])^{1/2}}\left[r\eta b N(1 - \|\boldsymbol{\rho}\|^2) - \boldsymbol{\rho}^\top\boldsymbol{\psi}\right] > 0\,,$$

Theorem 4.2 yields existence of a unique $\hat{p} > 0$ solving $L(p) = 0$, and this is the optimal premium rate. If further $b = 0$, then there is a closed form solution for $\hat{p}$, and the elasticity Φ depends only on θ and $\frac{r\eta}{\gamma}$,

$$\Phi = \frac{\frac{r\eta}{\gamma}\sqrt{\theta(1+\theta)}}{\frac{r\eta}{\gamma}\sqrt{\theta(1+\theta)} - \left(\frac{1}{1-\frac{r\eta}{\gamma}}\right)^\theta + 1}\,.$$

In this case, the function $\Theta(z)$ in (A.2) can be rewritten as

$$\begin{aligned}\Theta(z) &= N\lambda_0 \exp(-\Phi)\,[1+\Phi]\,[m_Y(z) - 1] \\ &\quad - \frac{z}{r\eta}\left\{r - \delta + \frac{1}{2}\|\boldsymbol{\psi}\|^2 + N\lambda_0\exp(-\Phi)\,[1+\Phi]\,[m_Y(r\eta) - 1]\right\} + \frac{z^2}{2(r\eta)^2}\|\boldsymbol{\psi}\|^2\,.\end{aligned}$$

If we further assume that the insurance company only reinvests any surplus at the risk-free rate, $\hat{\mathbf{A}} = \mathbf{0}$, and $\delta = r$, then it suffices to find z^* solving $\tilde{\Theta}(z) = 0$, for

$$\tilde{\Theta}(z) = [m_Y(z) - 1] - \frac{z}{r\eta}[m_Y(r\eta) - 1] .$$

No closed-form solution for z^* is available, but it clearly depends only on the distribution of claim sizes Y and $r\eta$. Note that $\tilde{\Theta}(z)$ is well-defined only on the domain of $m_Y(z)$, so $\frac{1}{w_0}\log\frac{1}{\varepsilon}$ must belong to this, too. Moreover, by implicit differentiation,

$$\frac{\partial z^*}{\partial \eta} = \left.\frac{-\frac{\partial\tilde{\Theta}}{\partial\eta}}{\frac{\partial\tilde{\Theta}}{\partial z}}\right|_{z=z^*} = \frac{\frac{z^*}{r\eta^2}\left\{1 - m_Y(r\eta) + r\eta m_Y'(r\eta)\right\}}{m_Y(z^*) - \frac{1}{r\eta}[m_Y(r\eta) - 1]} .$$

The denominator is positive since $\tilde{\Theta}(z^*) = 0$ implies that $\tilde{\Theta}$ is increasing at z^*, for fixed η. Therefore, z^* increases with η if $1 - m_Y(r\eta) + r\eta m_Y'(r\eta) > 0$, which holds for $\frac{r\eta}{\gamma} < 1$ and

$$\frac{r\eta}{\gamma}\sqrt{\theta(1+\theta)} + 1 > \left(\frac{1}{1 - \frac{r\eta}{\gamma}}\right)^{\theta} .$$

Furthermore, the conditions $z^* < \gamma$ and $z^* \geq \frac{1}{w_0}\log\frac{1}{\varepsilon}$ together imply

$$\gamma > \frac{1}{w_0}\log\frac{1}{\varepsilon} ,$$

i.e., $\varepsilon > e^{-w_0\gamma}$, so ε cannot be too small. In addition, using Young's inequality for products, it can be proved that $1 - m_Y(r\eta) + r\eta m_Y'(r\eta) > 0$ for the gamma distribution, so z^* does increase with η if $\delta = r$ in this case. Returning to the special case $\theta = 1$, i.e., $Y \sim \mathrm{Exp}(\gamma)$, we arrive at the following sharpening of Corollary A.1, complementing Theorem 4.2 for the unperturbed case.

Corollary A.2. *If the interest rate, company risk aversion, and mean claim satisfy*

$$\frac{r\eta}{\gamma} < 1 - \frac{1}{\sqrt{2}} ,$$

then the probability of ultimate ruin in case of exponentially distributed claim sizes, $\delta = r$, $b = 0$, and fixed deductible, $\bar{K} = 0$, is given by

$$\hat{\Psi}(w_0) = \left(1 - \frac{r\eta}{\gamma}\right) e^{-r\eta w_0} .$$

B Asset prices with jump risk

Following Aït-Sahalia et al. (2009), the price dynamics for risky assets in (2.3) can be extended to a mixture of continuous and jump processes,

$$\mathrm{d}S_t^i = S_{t-}^i \Big[\mu^i \,\mathrm{d}t + \sum_{j=1}^{I} \sigma^{ij} \,\mathrm{d}B_t^j + \bar{\nu}_i \mathrm{d}J_t\Big]\,, \;\; S_0^i > 0\,, \quad i = 1, \ldots, I\,. \tag{B.1}$$

As in (2.3), $\mathbf{B}_t = (B_t^1, \ldots, B_t^I)^\top$ is an I-dimensional vector of independent standard Brownian motions with respect to $\mathbb{F}$, $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^I)^\top \in \mathbb{R}^I$ is the vector of conditional expected returns, given no jumps, and $\boldsymbol{\sigma} = \left(\sigma^{ij}\right)_{1 \le i,j \le I} \in \mathbb{R}^{I \times I}$ is the volatility or diffusion matrix, with associated variance-covariance matrix $\boldsymbol{\Sigma} := \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$. Further, J_t represents an economy-wide jump in asset prices, modeled by the compound Poisson process

$$J_t = \sum_{m=1}^{\bar{N}_t} Z_m,$$

where $\bar{N}_t$ is a scalar Poisson process with intensity $\bar{\lambda}$, Z_m are i.i.d. random jump sizes with measure $H(\mathrm{d}z)$, independent of $\left(\bar{N}_t\right)_{t \ge 0}$, and $\bar{\nu}_i \in (-1, 1)$ are asset-specific loadings on the common jumps. Let $\bar{\boldsymbol{\nu}} = (\bar{\nu}_1, \ldots, \bar{\nu}_I)^\top$. If $\bar{\lambda} = 0$, the model for the risky assets collapses to the standard Black and Scholes (1973) model in (2.3).

With risky assets following the jump-diffusion process in (B.1), the budget constraint of the insurance company is given by the controlled SDE

$$\begin{aligned}\mathrm{d}W_t^{c,p,K,\boldsymbol{A}} &= [rW_t - c_t]\,\mathrm{d}t + \mathbf{A}_t^\top \left[(\boldsymbol{\mu} - r\underline{\mathbf{1}})\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t + \bar{\boldsymbol{\nu}}\mathrm{d}J_t\right] \\ &\quad + n\left(p_t, K_t\right)\left(p_t \,\mathrm{d}t - b\,\mathrm{d}\overline{B}_t\right) - \int_{\mathbb{R}_+} (y - K_t)^+ N^{p,K}(\mathrm{d}y, \mathrm{d}t)\,, \quad W_0^{c,p,K,\boldsymbol{A}} = w_0\,,\end{aligned}$$

generalizing (2.7), and the HJB equation reads

$$-\delta\vartheta(w) + \sup_{(c,p,K,\mathbf{A}) \in \mathcal{A}} \left\{U(c) + [\mathcal{L}^{c,p,K,\mathbf{A}}\vartheta](w)\right\} = 0\,,$$

where $\mathcal{L}^{c,p,K,\mathbf{A}}$ is the augmented operator

$$\begin{aligned}[\mathcal{L}^{c,p,K,\mathbf{A}}\vartheta](w) &= \left[rw + \mathbf{A}^\top(\boldsymbol{\mu} - r\underline{\mathbf{1}}) + n(p,K)p - c\right]\vartheta'(w) \\ &\quad + \frac{1}{2}\left[\left\|\boldsymbol{\sigma}^\top\mathbf{A}\right\|^2 + n(p,K)^2 b^2 - 2n(p,K)b\mathbf{A}^\top\boldsymbol{\sigma}\boldsymbol{\rho}\right]\vartheta''(w) \\ &\quad + n(p,K)\lambda(p,K)\left[\int_0^\infty \vartheta(w - (y-K)^+)\,F(\mathrm{d}y) - \vartheta(w)\right] \\ &\quad + \bar{\lambda}\left[\int \vartheta(w + \mathbf{A}^\top\bar{\boldsymbol{\nu}}z)G(\mathrm{d}z) - \vartheta(w)\right],\end{aligned}$$

generalizing (3.3).

As in the pure diffusion case, we conjecture that a solution to the HJB equation takes the form

$$\vartheta(w) = \beta e^{-r\eta w}\,, \quad \beta < 0\,. \tag{B.2}$$

Finding an interior solution to the HJB equation for this conjecture requires maximizing

$$U(c) + cr\eta\beta e^{-r\eta w} \tag{B.3}$$

over $c \in \mathbb{R}$, and

$$\begin{aligned} & r\eta[n(p,K)p + \mathbf{A}^\top(\boldsymbol{\mu} - r\underline{\mathbf{1}})] \\ & \qquad - \frac{(r\eta)^2}{2}\Big[\left\|\boldsymbol{\sigma}^\top\mathbf{A}\right\|^2 + n(p,K)^2b^2 - 2n(p,K)b\mathbf{A}^\top\boldsymbol{\sigma}\boldsymbol{\rho}\Big] \\ & \qquad\qquad - n(p,K)\lambda(p,K)q_1(K) - \bar{\lambda}q_2(\mathbf{A}) \end{aligned} \tag{B.4}$$

over $(p, K, \mathbf{A}) \in \mathbb{R}_+^2 \times \mathbb{R}^I$, with $q_1(K) = q(K)$ from (3.7), and

$$q_2(\mathbf{A}) := \mathbb{E}\left[e^{-r\eta\mathbf{A}^\top\bar{\boldsymbol{\nu}}z}\right] - 1 = \int_0^\infty e^{-r\eta\mathbf{A}^\top\bar{\boldsymbol{\nu}}z}H(\mathrm{d}z) - 1\,. \tag{B.5}$$

The first-order necessary optimality conditions for maximization with respect to c and $\mathbf{A}$ are, respectively,

$$e^{-\eta c} + r\eta\beta e^{-r\eta w} = 0\,, \tag{B.6}$$

$$\mu_i - r - r\eta([\boldsymbol{\sigma}\boldsymbol{\sigma}^\top\mathbf{A}]_i - n(p,K)b[\boldsymbol{\sigma}\boldsymbol{\rho}]_i) + \bar{\lambda}\nu_i q_3(\mathbf{A}) = 0\,, \quad i = 1,\dots,I\,, \tag{B.7}$$

with

$$q_3(\mathbf{A}) := \mathbb{E}\left[ze^{-r\eta\mathbf{A}^\top\bar{\boldsymbol{\nu}}z}\right] = \int_0^\infty ze^{-r\eta\mathbf{A}^\top\bar{\boldsymbol{\nu}}z}H(\mathrm{d}z)\,. \tag{B.8}$$

Solving the system (B.7) yields optimal amounts of financial wealth to be invested in risky assets, as functions of the premium rate and deductible, $\hat{\mathbf{A}}(p, K)$. Explicit solutions can be obtained under particular assumptions on the distribution of jump sizes that imply (i) that the function in (B.4) is concave, and (ii) convergent integrals in (B.5) and (B.8). For example, Merton (1976) and Das and Uppal (2004) assume log-normality, whereas Ait-Sahalia and Matthys (2019) assume an exponential distribution.

Solving (B.6) yields a solution for optimal consumption identical to that in (3.10), $\hat{c}_t = \hat{c}(W_t)$. Again, optimal consumption is affine in wealth, positive for wealth above a suitable threshold $w^\star$ of the form (3.15), and the condition $w^\star \geq 0$ ruling out the event of positive consumption coinciding with negative wealth takes the form

$$\delta \leq r - \tilde{Q}(\hat{p}, \hat{K})\,.$$

Here, $\tilde{Q}(\hat{p}, \hat{K})$ is the jump-augmented profile Hamiltonian for (p, K), obtained by substitution of the optimal investment strategy $\hat{\mathbf{A}}(p, K)$ into (B.4),

$$\begin{aligned} \widetilde{Q}(p,K) &:= r\eta[n(p,K)p + \hat{\mathbf{A}}(p,K)^\top(\boldsymbol{\mu} - r\underline{\mathbf{1}})] \\ & \qquad - \frac{(r\eta)^2}{2}\Big[\left\|\boldsymbol{\sigma}^\top\hat{\mathbf{A}}(p,K)\right\|^2 + n(p,K)^2b^2 - 2n(p,K)b\hat{\mathbf{A}}(p,K)^\top\boldsymbol{\sigma}\boldsymbol{\rho}\Big] \\ & \qquad\qquad - n(p,K)\lambda(p,K)q_1(K) - \bar{\lambda}q_2(\hat{\mathbf{A}}(p,K))\,. \end{aligned} \tag{B.9}$$

The optimal premium $\hat{p}$ and deductible $\hat{K}$ are determined by maximization of (B.9) over $(p, K) \in \mathbb{R}_+^2$. The conjecture (B.2) for the value function is verified using the same approach as in the proof of Proposition (3.1), with optimal value of β in the presence of jumps given by

$$\hat{\beta} = -\frac{1}{r\eta} \exp\left(1 - \frac{1}{r}\left[\delta + \widetilde{Q}(\hat{p}, \hat{K})\right]\right) .$$

C Average arrival rate of claims under partial information

Let $1/S$ have a gamma distribution with density $\frac{\zeta^\tau}{\Gamma(\tau)} y^{\tau-1} e^{-\zeta y}$ and let Λ be exponentially distributed with $\mathbb{E}[\Lambda] = \lambda_0$. The law of total probability yields

$$\mathbb{E}[\Lambda \,|\, S\Lambda > x] = \frac{\mathbb{E}[\Lambda \mathbf{1}_{\{S\Lambda > x\}}]}{\mathbb{P}(S\Lambda > x)} = \frac{\int_0^\infty \mathbb{E}[\Lambda \mathbf{1}_{\{\Lambda > x/S\}} \,|\, S = s] f_S(s)\, ds}{\mathbb{P}(S\Lambda > x)} .$$

Using the moment generating function of $1/S$, we get

$$\mathbb{P}(S\Lambda > x) = \mathbb{P}\Big(\Lambda > \frac{x}{S}\Big) = \mathbb{E}\Big[e^{-\frac{x}{\lambda_0}\frac{1}{S}}\Big] = \left(\frac{\zeta}{\zeta + \frac{x}{\lambda_0}}\right)^\tau ,$$

and thus at $x = \alpha(K)p$ the demand function (5.1). Now observe that

$$\begin{aligned} \mathbb{E}[\Lambda \mathbf{1}_{\{\Lambda > x/S\}} \,|\, S = s] &= \mathbb{E}\Big[\Lambda \mathbf{1}_{\{\Lambda > \frac{x}{s}\}}\Big] \\ &= \mathbb{E}\Big[\Lambda \,|\, \Lambda > \frac{x}{s}\Big] \mathbb{P}\Big(\Lambda > \frac{x}{s}\Big) \\ &= \Big(\lambda_0 + \frac{x}{s}\Big) e^{-\frac{x}{\lambda_0 s}} . \end{aligned}$$

It follows that

$$\int_0^\infty \mathbb{E}[\Lambda \mathbf{1}_{\{\Lambda > x/S\}} \,|\, S = s] f_S(s)\, ds = \mathbb{E}\left[\Big(\lambda_0 + \frac{x}{S}\Big) e^{-\frac{x}{\lambda_0}\frac{1}{S}}\right] .$$

Using the change-of-variable $z = (\zeta + \frac{x}{\lambda_0})y$ and the identity $\Gamma(\tau+1) = \tau\Gamma(\tau)$, we get

$$\begin{aligned} \mathbb{E}\left[\frac{1}{S} e^{-\frac{x}{\lambda_0}\frac{1}{S}}\right] &= \frac{\zeta^\tau}{\Gamma(\tau)} \int_0^\infty y e^{-\frac{x}{\lambda_0} y} y^{\tau-1} e^{-\zeta y}\, dy \\ &= \frac{\zeta^\tau}{\Gamma(\tau)} \frac{1}{(\zeta + \frac{x}{\lambda_0})^{1+\tau}} \int_0^\infty z^\tau e^{-z}\, dz \\ &= \frac{\tau \zeta^\tau}{(\zeta + \frac{x}{\lambda_0})^{1+\tau}} . \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[\Lambda \,|\, S\Lambda > x] &= \lambda_0 + x \frac{\tau \zeta^\tau}{(\zeta + \frac{x}{\lambda_0})^{1+\tau}} \left(\frac{\zeta}{\zeta + \frac{x}{\lambda_0}}\right)^{-\tau} \\ &= \lambda_0 + \frac{x\tau}{\zeta + \frac{x}{\lambda_0}} , \end{aligned}$$

which at $x = \alpha(K)p$ is the average arrival rate of claims in (5.2).

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