Supplement to "Peso problems in the estimation of the C-CAPM"

(*Quantitative Economics*, Vol. 13, No. 1, January 2022, 259–313)

Juan Carlos Parra-Alvarez
Department of Economics and Business Economics, Aarhus University, CREATES, and Danish Finance Institute

Olaf Posch
Department of Economics, Universität Hamburg and CREATES

Andreas Schrimpf
Bank for International Settlements, CEPR

Appendix B

B.1 *General equilibrium prices in the endowment economy*

We use the stochastic differential for consumption implied by the Euler equation (39) and the market clearing condition $C_t = Y_t$ together with the exogenous dividend process (7).

Proposition B.1 (Asset pricing). *In general equilibrium, market clearing implies*

$$\begin{aligned}
\mu_M - r &= -\frac{u''(C_t)C_W W_t}{u'\big(C(W_t)\big)}\sigma_M^2 - \frac{u'\big(e^{\bar{\nu}}C(W_t)\big)}{u'\big(C(W_t)\big)}\big((1-e^{\kappa})q - \zeta_M\big)\lambda, \\
\sigma_M &= \bar{\sigma} C_t/(C_W W_t), \\
r &= \rho - \frac{u''(C_t)C_t}{u'(C_t)}\bar{\mu} - \frac{1}{2}\frac{u'''(C_t)C_t^2}{u'(C_t)}\bar{\sigma}^2 + \lambda - \big(1-\big(1-e^{\kappa}\big)q\big)\frac{u'\big(e^{\bar{\nu}}C_t\big)}{u'(C_t)}\lambda,
\end{aligned}$$

as well as implicitly the portfolio jump-size

$$C\big(\big(1-\zeta_M(t)\big)W_t\big) = \exp(\bar{\nu})C(W_t).$$

Proof. Using the inverse function, we are able to determine the path for consumption ($u'' \neq 0$). From the Euler equation (39), we obtain

$$dC_t = \Big((\rho - \mu_M + \lambda)u'(C_t)/u''(C_t) - \sigma_M^2 W_t C_W - \frac{1}{2}u'''(C_t)/u''(C_t)\sigma_M^2 W_t^2 C_W^2$$

Juan Carlos Parra-Alvarez: jcpalva@business.aau.dk
Olaf Posch: olaf.posch@uni-hamburg.de
Andreas Schrimpf: andreas.schrimpf@bis.org

Available at http://qeconomics.org. https://doi.org/10.3982/QE1478

$$
\begin{aligned}
&- E^{\zeta}\big[u'\big(C\big((1-\zeta_M(t))W_t\big)\big)\big(1-\zeta_M(t)\big)\big]\lambda/u''(C_t)\Big)\,dt \\
&+ \sigma_M W_t C_W\,dB_t + \big(C\big((1-\zeta_M(t))W_{t-}\big) - C(W_{t-})\big)\,dN_t,
\end{aligned}
\tag{B.1}
$$

where we employed the inverse function $c = g(u'(c))$, which has

$$
g'\big(u'(c)\big) = 1/u''(c), \qquad g''\big(u'(c)\big) = -u'''(c)/\big(u''(c)\big)^3.
$$

Economically, concave utility ($u'(c) > 0$, $u''(c) < 0$) implies risk aversion, whereas convex marginal utility, $u'''(c) > 0$, implies a positive precautionary saving motive. Accordingly, $-u''(c)/u'(c)$ measures absolute risk aversion, whereas $-u'''(c)/u''(c)$ measures the degree of absolute prudence, that is, the intensity of the precautionary saving motive.

Because output is perishable, using the market clearing condition $Y_t = C_t = A_t$, and

$$
dC_t = \bar{\mu} C_t\,dt + \bar{\sigma} C_t\,dB_t + \big(\exp(\bar{\nu}) - 1\big)C_{t-}\,dN_t,
$$

the parameters of price dynamics are pinned down in general equilibrium. In particular, we obtain J_t implicitly as function of $\bar{\nu}$, D_t, and the curvature of the consumption function, where $\tilde{C}(W_t) \equiv C((1-\zeta_M(t))W_t)/C(W_t)$ defines optimal consumption jumps. For market clearing, we require the percentage jump in aggregate consumption to match the size of the disaster, $\exp(\bar{\nu}) = \tilde{C}(W_t)$, and thus $\exp(\bar{\nu}) = C((1+(J_t - D_t)w_t + D_t)W_t)/C(W_t)$ implies a constant jump size. For consumption being linear homogeneous in wealth,

$$
\zeta_M = e^{\bar{\nu}} - 1.
$$

Similarly, the market clearing condition pins down $\sigma_M W_t C_W = \bar{\sigma} C_t$, and

$$
\mu_M - r = -\frac{u''(C_t) C_W W_t}{u'\big(C(W_t)\big)}\sigma_M^2 - \frac{u'\big(e^{\bar{\nu}} C(W_t)\big)}{u'\big(C(W_t)\big)}\big((1-e^{\kappa})q - \zeta_M\big)\lambda.
$$

Inserting our results back into (B.1), we obtain that consumption follows

$$
\begin{aligned}
dC_t = {} & (\rho - r + \lambda)\frac{u'(C_t)}{u''(C_t)}\,dt - \frac{1}{2}\frac{u'''(C_t)}{u''(C_t)}\sigma_M^2 W_t^2 C_W^2\,dt \\
& - \big(1 - \big(1-e^{\kappa}\big)q\big)\frac{u'\big(e^{\bar{\nu}} C_t\big)}{u''(C_t)}\lambda\,dt \\
& + \sigma_M W_t C_W\,dB_t + \big(C\big((1-\zeta_M(t))W_{t-}\big) - C(W_{t-})\big)\,dN_t.
\end{aligned}
$$

This in turn determines the return on the riskless asset

$$
r = \rho - \frac{u''(C_t)C_t}{u'(C_t)}\bar{\mu} - \frac{1}{2}\frac{u'''(C_t)C_t^2}{u'(C_t)}\bar{\sigma}^2 + \lambda - \big(1 - \big(1-e^{\kappa}\big)q\big)\frac{u'\big(e^{\bar{\nu}} C_t\big)}{u'(C_t)}\lambda.
$$

As a result, the higher the subjective rate of time preference, ρ, the higher is the general equilibrium interest rate to induce individuals to defer consumption (cf. Breeden (1986)). For convex marginal utility (decreasing absolute risk aversion), $u'''(c) > 0$, a lower conditional variance of dividend growth, $\bar{\sigma}^2$, and a higher conditional mean of dividend growth, $\bar{\mu}$, and a higher default probability, q, decrease the bond price and increases the interest rate. □

PROPOSITION B.2 (PDE approach). *An alternative characterization of the no-arbitrage condition is given by the PDE*

$$E_t\left[\frac{d(m_t P_t^d)}{m_t P_t^d}\right] + \frac{C_t}{P_t^d}\,dt = 0.$$

PROOF. By application of Itô's formula

$$\begin{aligned} d(m_t P_t^d) &= \big(dP_t^d - (e^{\bar{\nu}} - 1)P_{t-}^d\,dN_t\big)m_t + \big(dm_t - (e^{-\gamma\bar{\nu}} - 1)m_{t-}\,dN_t\big)P_t^d + dm_t\,dP_t^d \\ &\quad + \big(e^{(1-\gamma)\bar{\nu}} - 1\big)m_{t-}P_{t-}^d\,dN_t \end{aligned}$$

such that

$$\begin{aligned} E_t\big[d(m_t P_t^d)\big] &= E_t\big[dP_t^d\big]m_t + E_t[dm_t]P_t^d + E_t\big[dm_t\,dP_t^d\big] \\ &\quad + \big(-(e^{\bar{\nu}} - 1) - (e^{-\gamma\bar{\nu}} - 1) + (e^{(1-\gamma)\bar{\nu}} - 1)\big)\lambda P_t^d m_t\,dt \end{aligned}$$

and thus the instantaneous return to the asset in (48) is

$$\frac{1}{dt}E_t\big[dR_t^d\big] = r_t^f - \frac{1}{dt}E_t\left[\frac{dm_t\,dP_t^d}{m_t P_t^d}\right] - \big((e^{(1-\gamma)\bar{\nu}} - 1) - (e^{\bar{\nu}} - 1) - (e^{-\gamma\bar{\nu}} - 1)\big)\lambda,$$

where we defined $dR_t^d \equiv (dP_t^d)/P_t^d + (A_t/P_t^d)\,dt$. Inserting the solution in (49) yields

$$\begin{aligned} &-\big(r - (1 - e^{\kappa})e^{-\gamma\bar{\nu}}q\lambda + (e^{-\gamma\bar{\nu}} - 1)\lambda\big) + (e^{-\gamma\bar{\nu}} - 1)\lambda + \bar{\mu} + (e^{\bar{\nu}} - 1)\lambda - \gamma\bar{\sigma}^2 \\ &\quad + \big((e^{(1-\gamma)\bar{\nu}} - 1) - (e^{\bar{\nu}} - 1) - (e^{-\gamma\bar{\nu}} - 1)\big)\lambda \\ &\quad + \rho - (1-\gamma)\left(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2\right) - \frac{1}{2}\big((1-\gamma)\bar{\sigma}\big)^2 - \big(1 - e^{(1-\gamma)\bar{\nu}}\big)\lambda = 0 \end{aligned}$$

and by collecting terms yields back the equilibrium interest rate

$$\begin{aligned} r &= \rho + \gamma\bar{\mu} - \gamma\bar{\sigma}^2 - \big(e^{-\gamma\bar{\nu}} - 1\big)\lambda \\ &\quad + (1-\gamma)\frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\big((1-\gamma)\bar{\sigma}\big)^2 + \big(1 - e^{\kappa}\big)qe^{-\gamma\bar{\nu}}\lambda \\ &= \rho + \gamma\bar{\mu} - \frac{3}{2}\gamma\bar{\sigma}^2 + \frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\big(1 - 2\gamma + \gamma^2\big)\bar{\sigma}^2 + \lambda - \big(1 - (1 - e^{\kappa})q\big)e^{-\gamma\bar{\nu}}\lambda \\ &= \rho + \gamma\bar{\mu} - \frac{1}{2}\gamma(1+\gamma)\bar{\sigma}^2 + \lambda - \big(1 - (1 - e^{\kappa})q\big)e^{-\gamma\bar{\nu}}\lambda, \end{aligned}$$

which completes the proof that the PDE approach gives the same price P_t^d. □

B.2 *An alternative mimicking economy with rare events*

B.2.1 *The underlying production economy* Consider the representative-agent neoclassical production economy in Appendix A.3. The following propositions show the optimal consumption function, the SDF, and the equilibrium prices for different asset classes, for the parametric restriction $\alpha = \gamma$.

Proposition B.3 (Linear-policy function). *Suppose the production function $F(K_t, L)$ is $Y_t = A_t K_t^\alpha L^{1-\alpha}$, utility has constant relative risk aversion, that is, $-u''(C_t)C_t/u'(C_t) = \gamma$, and let $\alpha = \gamma$ (with $\gamma < 1$). Then optimal consumption is linear in wealth.*

$$\begin{aligned} &\alpha = \gamma \quad \Rightarrow \quad C_t = C(W_t) = kW_t, \\ &k \equiv \big(\rho - \big(e^{(1-\gamma)\nu} - 1\big)\lambda + (1-\gamma)\delta\big)/\gamma + \frac{1}{2}(1-\gamma)\sigma^2, \end{aligned} \tag{B.2}$$

where k denotes the marginal propensity to consume out of (physical) wealth.

Proof. The idea of the proof follows closely that of Proposition A.9. An educated guess of the value function is

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\gamma}}{1-\gamma} + f(A_t). \tag{B.3}$$

From (55), optimal consumption is a constant fraction of wealth,

$$C_t^{-\gamma} = \mathbb{C}_1 W_t^{-\gamma} \quad \Leftrightarrow \quad C_t = \mathbb{C}_1^{-1/\gamma} W_t.$$

Now use the maximized Bellman equation (56), the property of the Cobb–Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^\alpha L_t^{-\alpha}$, together with the transformation $K_t \equiv L W_t$, and insert the solution candidate to obtain

$$\begin{aligned} \rho \frac{\mathbb{C}_1 W_t^{1-\gamma}}{1-\gamma} = {} & \frac{\mathbb{C}_1^{-\frac{1-\gamma}{\gamma}} W_t^{1-\gamma}}{1-\gamma} + \big(\alpha A_t W_t^{\alpha-1} W_t - \delta W_t + (1-\alpha) A_t W_t^\alpha - \mathbb{C}_1^{-1/\gamma} W_t\big)\mathbb{C}_1 W_t^{-\gamma} \\ & - \frac{1}{2}\gamma \mathbb{C}_1 W_t^{1-\gamma}\sigma^2 - g(A_t) + \big(e^{(1-\gamma)\nu} - 1\big)\frac{\mathbb{C}_1 W_t^{1-\gamma}}{1-\gamma}\lambda, \end{aligned}$$

where we defined $g(A_t) \equiv \rho f(A_t) - f_A \bar{\mu} A_t - \frac{1}{2} f_{AA} \bar{\sigma}^2 A_t^2 - [f(e^{\bar{\nu}} A_t) - f(A_t)]\bar{\lambda}$. When imposing the condition $\alpha = \gamma$ and $g(A_t) = \mathbb{C}_1 A_t$ it can be simplified to

$$\begin{aligned} & \big(\rho - \big(e^{(1-\gamma)\nu} - 1\big)\lambda\big)\frac{\mathbb{C}_1 W_t^{1-\gamma}}{1-\gamma} + g(A_t) \\ & \quad = \frac{\mathbb{C}_1^{-\frac{1-\gamma}{\gamma}} W_t^{1-\gamma}}{1-\gamma} + \big(A_t W_t^{\alpha-\gamma} - \delta W_t^{1-\gamma} - \mathbb{C}_1^{-1/\gamma} W_t^{1-\gamma}\big)\mathbb{C}_1 - \frac{1}{2}\gamma \mathbb{C}_1 W_t^{1-\gamma}\sigma^2 \\ & \Leftrightarrow \quad \big(\rho - \big(e^{(1-\gamma)\nu} - 1\big)\lambda\big) W_t^{1-\gamma} = \gamma \mathbb{C}_1^{-1/\gamma} W_t^{1-\gamma} - (1-\gamma)\delta W_t^{1-\gamma} - \frac{1}{2}\gamma(1-\gamma) W_t^{1-\gamma}\sigma^2, \end{aligned}$$

which implies that $\mathbb{C}_1^{-1/\gamma} = (\rho - (e^{(1-\gamma)\nu} - 1)\lambda + (1-\gamma)\delta + \frac{1}{2}\gamma(1-\gamma)\sigma^2)/\gamma$. This proves that the guess (B.3) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal policy function for consumption. □

PROPOSITION B.4 (Rental rate of capital). *Suppose the production function $F(K_t, L)$ is $Y_t = A_t K_t^\alpha L^{1-\alpha}$. The rental rate of capital is obtained from the marginal product of capital, $r_t = \alpha A_t K_t^{\alpha-1}$, and follows the reducible stochastic differential equation,*

$$\begin{aligned} dr_t = c_1(c_2 - r_t)r_t\,dt + (\alpha-1)\sigma r_t\,dZ_t + \bar{\sigma} r_t\,d\bar{B}_t + \big(\exp\big((\alpha-1)\nu\big) - 1\big)r_{t-}\,dN_t \\ + \big(\exp(\bar{\nu}) - 1\big)r_{t-}\,d\bar{N}_t, \end{aligned} \tag{B.4}$$

in which the constants c_1 and c_2 for the parametric restriction $\alpha = \gamma$ are given by

$$c_1 \equiv \frac{1-\alpha}{\alpha}, \qquad c_2 \equiv \alpha k + \alpha\delta - \frac{1}{2}\alpha(\alpha-2)\sigma^2 - \frac{\alpha}{\alpha-1}\bar{\mu}.$$

PROOF. The idea of the proof is along the lines of Proposition A.10. □

PROPOSITION B.5 (Stochastic discount factor). *Following the assumptions in Proposition B.3, the stochastic discount factor (SDF) is given by*

$$m_s/m_t = e^{-\int_t^s (r_v - \delta)\,dv + [\lambda - e^{(1-\gamma)\nu}\lambda + \gamma\sigma^2 - \frac{1}{2}(\gamma\sigma)^2](s-t) - \gamma\sigma(Z_s - Z_t) - \gamma\nu(N_s - N_t)}. \tag{B.5}$$

PROOF. The idea of the proof is along the lines of Proposition A.11. □

PROPOSITION B.6 (Risky bond). *Consider a risky asset that pays at the rate r_t in $t+1$. The one-period holding return of an asset with the random payoff $X_{b,t+1} = e^{\int_t^{t+1} r_s\,ds}$ is*

$$R_{t+1}^b = \exp\bigg(\int_t^{t+1} \big(r_v - \delta - \gamma\sigma^2 - e^{-\gamma\nu}\big(1 - e^\nu\big)\lambda\big)\,dv\bigg). \tag{B.6}$$

PROOF. Substitute the random payoff $X_{b,t+1}$ in (2) to obtain the equilibrium price of this risky bond at time t as

$$P_t^b = E_t\bigg[\frac{m_{t+1}}{m_t} e^{\int_t^{t+1} r_s\,ds}\bigg].$$

Using the definition of the SDF (B.5) and making use of Lemma A.2 yields

$$P_t^b = e^{\delta + \gamma\sigma^2 + e^{-\gamma\nu}\lambda - e^{(1-\gamma)\nu}\lambda}.$$

For any $s > t$, $R_s^b = X_{b,s}/P_t^b$ denotes the gross return on the risky bond. The desired result follows by setting $s = t+1$. □

PROPOSITION B.7 (Risky asset). *The one-period holding return on an asset that pays one unit of output $X_{c,t+1} = A_{t+1}K_{t+1}^{\alpha}$ is*

$$R_s^c = \exp\left(\int_t^s \left(r_v - \delta - \frac{1}{2}\bar{\sigma}^2 - \lambda + e^{(1-\gamma)\nu}\lambda - \gamma\sigma^2 + \frac{1}{2}(\gamma\sigma)^2 - \left(e^{\bar{\nu}} - 1\right)\bar{\lambda}\right) dv\right)$$
$$\times \exp\left(\bar{\sigma}(\bar{B}_s - \bar{B}_t) + \alpha\sigma(Z_s - Z_t) + \alpha\nu(N_s - N_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t)\right). \tag{B.7}$$

PROOF. For any $s > t$, it follows from (62) and (63) that

$$A_s K_s^{\alpha}$$
$$= A_t K_t^{\alpha} e^{(\bar{\mu}-\frac{1}{2}\bar{\sigma}^2)(s-t)+\int_t^s (r_v - \alpha C_v/K_v - \alpha\delta - \alpha\frac{1}{2}\sigma^2)\, dv + \bar{\sigma}(B_s - B_t) + \alpha\sigma(Z_s - Z_t) + \alpha\nu(N_s - N_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t)}.$$

Set $s = t+1$ and substitute the payoff $X_{c,t+1}$ together with the definition of the SDF (B.5) into (2). Making use of Lemma A.2 compute the equilibrium price of this asset as

$$P_t^c = E_t\left[\frac{m_{t+1}}{m_t} A_{t+1} K_{t+1}^{\alpha}\right]$$
$$\Rightarrow \quad P_t^c = E_t\left[A_t K_t^{\alpha} e^{\bar{\mu} - \frac{1}{2}\bar{\sigma}^2 - \alpha k - \alpha\delta - \alpha\frac{1}{2}\sigma^2 + \delta + \lambda - e^{(1-\gamma)\nu}\lambda + \gamma\sigma^2 - \frac{1}{2}(\gamma\sigma)^2 + \bar{\sigma}(\bar{B}_{t+1} - \bar{B}_t) + \bar{\nu}(\bar{N}_{t+1} - \bar{N}_t)}\right]$$
$$= A_t K_t^{\alpha} e^{-(\alpha k + \alpha\delta + \alpha\frac{1}{2}\sigma^2 - \delta - \lambda + e^{(1-\gamma)\nu}\lambda - \gamma\sigma^2 + \frac{1}{2}(\gamma\sigma)^2 - \bar{\mu} - (e^{\bar{\nu}} - 1)\bar{\lambda})}.$$

For any $s > t$, $R_s^c = X_{c,s}/P_t^b$ denotes the gross return on the risky bond. The desired result follows by setting $s = t+1$. □

B.2.2 *Euler equation errors for $\alpha = \gamma$* Consider two assets, that is, the risky bond, R_{t+1}^b, and the risky claim on output, R_{t+1}^c. From the definition of Euler equation errors (3), for any asset i and CRRA preferences

$$e_R^i = E_t\left[e^{-\int_t^{t+1}(r_s - \delta)\, ds + \lambda - e^{(1-\gamma)\nu}\lambda + \gamma\sigma^2 - \frac{1}{2}(\gamma\sigma)^2 - \gamma\sigma(Z_{t+1} - Z_t) - \gamma\nu(N_{t+1} - N_t)} R_{t+1}^i\right] - 1,$$

where we inserted the SDFs from (B.5). Inserting the one-period holding equilibrium returns for the risky bond (B.6) yields

$$e_R^b = E_t\left[e^{(1 - e^{-\gamma\nu})\lambda - \frac{1}{2}(\gamma\sigma)^2 - \gamma\sigma(Z_{t+1} - Z_t) - \gamma\nu(N_{t+1} - N_t)}\right] - 1.$$

Conditional on no disasters, on average we can rationalize Euler equation errors

$$e^b_{R|N_{t+1} - N_t = 0} = \exp\left(\left(1 - e^{-\gamma\nu}\right)\lambda\right) - 1,$$

or, conditional on no rare events, on average we can rationalize Euler equation errors

$$e^b_{R|N_{t+1} - N_t = \bar{N}_{t+1} - \bar{N}_t = 0} = \exp\left(\left(1 - e^{-\gamma\nu}\right)\lambda\right) - 1.$$

Similarly, inserting the return on the claims on output (B.7) we obtain

$$e_R^c = E_t\left[e^{-\frac{1}{2}\bar{\sigma}^2 - (e^{\bar{\nu}} - 1)\bar{\lambda} + \bar{\sigma}(\bar{B}_{t+1} - \bar{B}_t) + \bar{\nu}(\bar{N}_{t+1} - \bar{N}_t)}\right] - 1.$$

EE errors based on excess returns are obtained from $e_X^i = e_R^i - e_R^b$ for any asset i.

B.2.3 *The mimicking endowment economy for $\alpha = \gamma$*

Technology Suppose production of perishable output, Y_t, is exogenously given: there is no possibility of affecting the output at any time. Let $Y_t = \alpha k A_t K_t^\alpha / r_t = k K_t$, where K_t is the aggregate capital stock, and A_t is stochastic technology or total factor productivity (TFP). Output is perishable. The law of motion of A_t is given in (50).

The capital stock is subject to stochastic depreciation,

$$dK_t = \big(A_t K_t^\alpha - (k+\delta)K_t\big)\,dt + \sigma K_t\,dZ_t + \big(\exp(\nu) - 1\big)K_{t-}\,dN_t, \tag{B.8}$$

in which Z_t is a standard Brownian motion (uncorrelated with $\bar{B}_t$), and N_t is a Poisson process with constant arrival rate λ.

Thus, in the mimicking endowment economy with $\alpha = \rho$, output follows

$$\begin{aligned} dY_t &= k\big(A_t K_t^\alpha - (k+\delta)K_t\big)\,dt + \sigma k K_t\,dZ_t \\ &\quad + \big(\exp(\nu) - 1\big)kK_{t-}\,dN_t \\ &= \big(A_t K_t^{\alpha-1} - (k+\delta)\big)Y_t\,dt + \sigma Y_t\,dZ_t \\ &\quad + \big(\exp(\nu) - 1\big)Y_{t-}\,dN_t \\ &= \big(r_t/\alpha - (k+\delta)\big)Y_t\,dt + \sigma Y_t\,dZ_t \\ &\quad + \big(\exp(\nu) - 1\big)Y_{t-}\,dN_t \\ &\equiv \mu_t Y_t\,dt + \sigma_t Y_t\,dZ_t + (Y_t - Y_{t-})\,dN_t \end{aligned}$$

with $\mu_t = r_t/\alpha - (k+\delta)$, $\sigma_t \equiv \sigma$ and $r_t = \alpha A_t K_t^{\alpha-1}$, such that

$$\begin{aligned} dr_t &= c_1(c_2 - r_t)r_t\,dt + (\alpha-1)\sigma r_t\,dZ_t + \bar{\sigma} r_t\,d\bar{B}_t \\ &\quad + \big(\exp\big((\alpha-1)\nu\big) - 1\big)r_{t-}\,dN_t \\ &\quad + \big(\exp(\bar{\nu}) - 1\big)r_{t-}\,d\bar{N}_t \end{aligned} \tag{B.9}$$

in which $c_1 \equiv \frac{1-\alpha}{\alpha}$, and $c_2 \equiv \alpha k + \alpha\delta - \frac{1}{2}\alpha(\alpha-2)\sigma^2 - \frac{\alpha}{\alpha-1}\bar{\mu}$.

Preferences The representative consumer maximizes expected discounted lifetime utility given in (8) and (9). Further assume that $1/\psi = \gamma$ such that the problem is reduced to the standard power utility case in (10).

Equilibrium In this economy, it is easy to determine equilibrium quantities and equilibrium asset holdings. The economy is closed, output will be consumed, $C_t = Y_t$, and households own the physical capital. Other assets are zero in net supply.

B.3 *Tables and figures*

See Figure B.1 and Tables B.1–B.8.

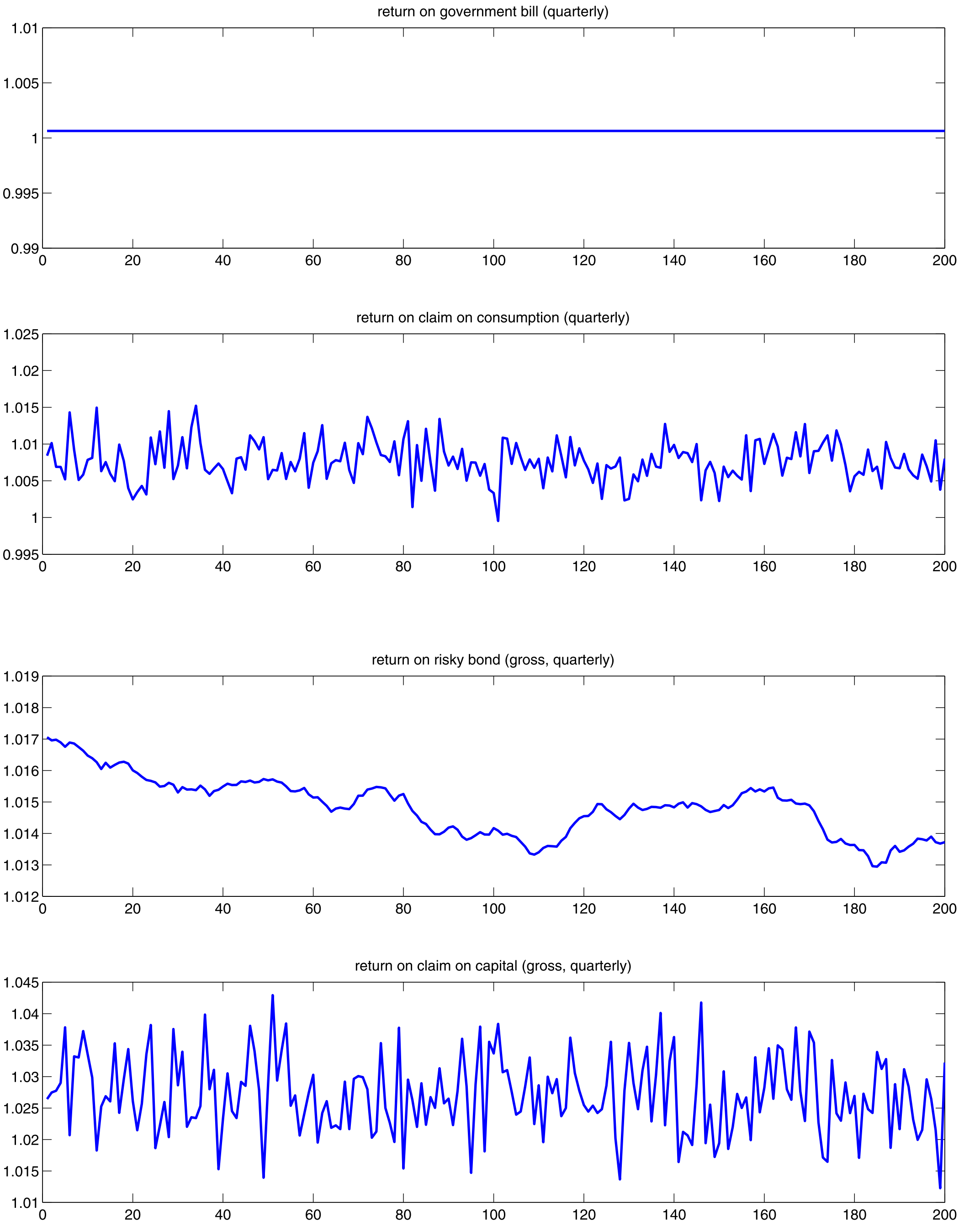


FIGURE B.1. General equilibrium asset returns. Notes: This figure illustrates the equilibrium asset returns and shows one realization of the return to the bonds and the risky assets in the simple endowment economy (upper two panels, parameterization (2) in Table B.1) and the endowment economy mimicking a production economy (lower two panels, parameterization (2) in Table B.2), respectively.

TABLE B.1. Robustness: simulation study (endowment economy).

		(1)	(2)	(3)	(4)
ρ	rate of time preference	0.03	0.03	**0.03**	**0.03**
γ	coef. of relative risk aversion	0.5	4	**4**	**4**
$\bar{\mu}$	consumption growth	0.01	0.01	**0.01**	**0.01**
$\bar{\sigma}$	consumption noise	0.005	0.005	**0.005**	**0.005**
$-\bar{\nu}$	size of consumption disaster	0.4	0.4	**0.4**	**0**
λ	consumption disaster probability	0.017	0.017	**0.017**	**0**
$-\kappa$	size of government default	0	0	**0.3**	**0**
q	default probability	0	0	**0.5**	**0**

TABLE B.2. Robustness: simulation study (production economy).

		(1)	(2)	(3)	(4)
ρ	rate of time preference	0.03	**0.024**	**0.016**	0.03
γ	coef. of relative risk aversion	0.5	**4**	**4**	4
α	output elasticity of capital	0.5	**0.6**	**0.6**	0.6
δ	capital depreciation	0.025	**0.025**	**0.025**	0.05
$\bar{\mu}$	productivity growth	0.02	**0.01**	**0.01**	0.01
$\bar{\sigma}$	productivity noise	0.01	**0.01**	**0.01**	0.01
$-\bar{\nu}$	size of productivity slump	0.01	**0.01**	**0**	0
$\bar{\lambda}$	productivity jump probability	0.2	**0.2**	**0**	0
σ	capital stochastic depreciation	0.005	**0.005**	**0.005**	0.005
$-\nu$	size of capital disaster	0.55	**0.55**	**0.55**	0
λ	capital disaster probability	0.017	**0.017**	**0.017**	0

TABLE B.3. Robustness: simulation study (long-run risk model).

		(1)	(2)	(3)	(4)
ρ	rate of time preference	**0.024**	0.024	**0.03**	0.02
γ	coef. of relative risk aversion	**10**	7.5	**10**	30
ψ	EIS	**1.5**	1.5	**1.5**	1.5
$\bar{\mu}$	consumption growth	**0.018**	0.018	**0.018**	0.018
κ_{μ}	LRR persistence	**0.256**	0.256	**0.3**	0.256
ν_{μ}	LRR volatility multiple	**0.528**	0.528	**0.456**	0.456
$\bar{\vartheta}$	baseline volatility ($\times 100$)	**0.0729**	0.0729	**0.0625**	0.0625
κ_{ϑ}	persistence volatility	**0.156**	0.156	**0.015**	0.156
ν_{ϑ}	vol-of-vol	**0.0035**	0.0035	**0.0027**	0.0027

TABLE B.4. C-CAPM simulation results (endowment economy). The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM observed at quarterly frequency in the endowment economy with rare events (cf. Section 3.1) for a parameterization as in column (3) in Table B.1; the bond return, the equity return, the equity premium, and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.97$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5000 Monte Carlo sample paths, each of length 50 years.

Results	Analytical solution parameterization (3)	Unconditional Mean	Std. dev.	Mode	Median
e_R^b	EE error risky bond	0.09	6.61	−5.55	−0.14
e_X^c	EE error excess return	−0.12	2.59	1.68	0.72
RMSE	root mean square error	3.86	3.20	4.05	3.98
Observed random variables					
R_{t+1}^b	bill return	1.16	0.36	1.35	1.35
R_{t+1}^c	equity return	2.49	0.62	3.04	2.45
$R_{t+1}^c - R_{t+1}^b$	equity premium	1.34	0.50	1.68	1.52
$\ln(C_{t+1}/C_t)$	consumption growth	0.33	0.75	0.98	0.27
Parameter estimates					
$\hat{\beta}$	factor of time preference	1.07	0.14	0.98	0.99
$\hat{\gamma}$	coef. of relative risk aversion	356.98	434.27	5.00	5.40
$\widehat{e_R^b}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e_X^c}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{\text{RMSE}}$	root mean square error	0.00	0.00	0.00	0.00

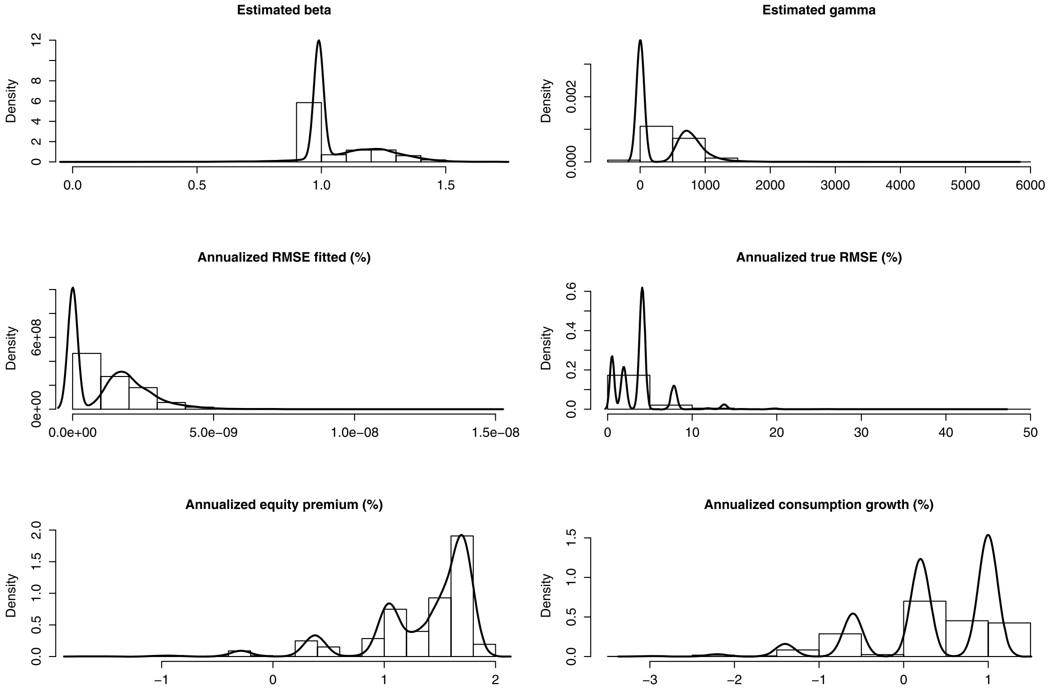


TABLE B.5. C-CAPM simulation results (production economy). The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM observed at quarterly frequency in the production economy with rare events (cf. Section 3.2) for a parameterization as in column (2) in Table B.2; the bond return, the equity return, the equity premium, and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5000 Monte Carlo sample paths, each of length 50 years.

Results	Constant-saving-function, parameterization (2)	Unconditional Mean	Std. dev.	Mode	Median
e_R^b	EE error risky bond	0.61	4.73	0.75	0.63
e_X^c	EE error excess return	−0.60	4.70	−0.75	−0.74
RMSE	root mean square error	3.69	3.00	0.68	4.36
Observed random variables					
R_{t+1}^b	bill return (gross)	7.40	1.10	6.39	7.21
R_{t+1}^c	equity return (gross)	10.74	0.77	10.82	10.86
$R_{t+1}^c - R_{t+1}^b$	equity premium	3.34	1.27	3.32	3.31
$\ln(C_{t+1}/C_t)$	consumption growth	1.80	0.43	1.71	1.84
Parameter estimates					
$\hat{\beta}$	factor of time preference	0.93	0.37	0.99	0.99
$\hat{\gamma}$	coef. of relative risk aversion	152.20	314.21	2.50	3.55
$\widehat{e_R^b}$	EE error risky bond	−0.01	0.02	0.00	0.00
$\widehat{e_X^c}$	EE excess return	1.14	1.78	0.00	0.00
$\widehat{\text{RMSE}}$	root mean square error	0.81	1.26	0.00	0.00

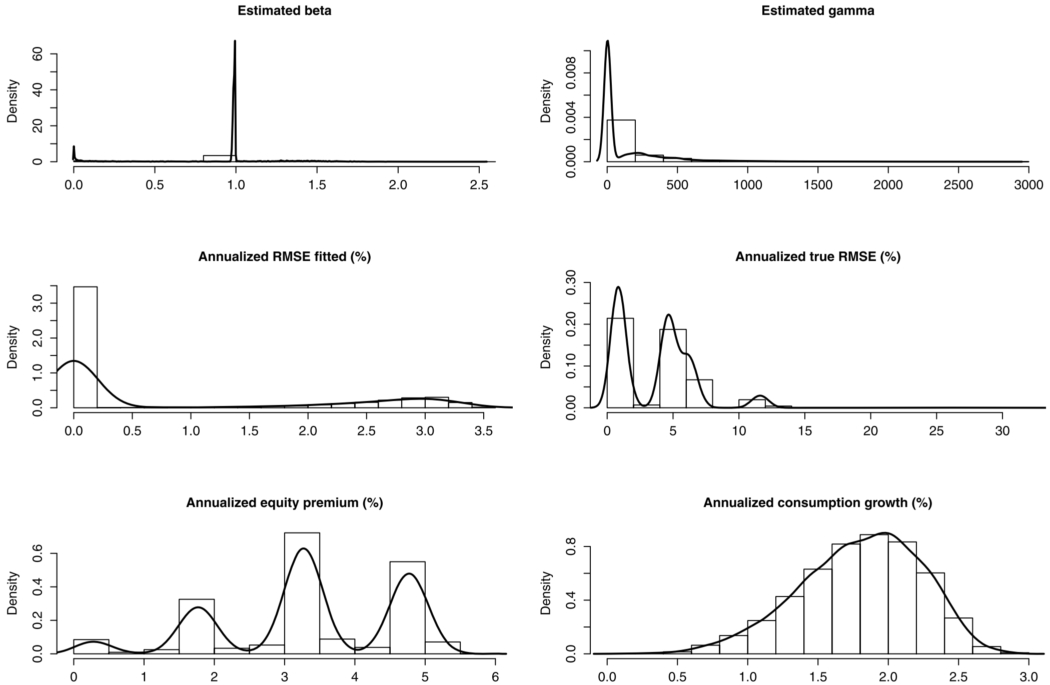


TABLE B.6. C-CAPM simulation results (production economy). The table reports the simulated Euler equation (EE) errors and RMSE (both annualized) for the standard C-CAPM observed at quarterly frequency in the production economy with rare events (cf. Section 3.2) for a parameterization as in column (3) in Table B.2; the bond return, the equity return, the equity premium, and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 4$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5000 Monte Carlo sample paths, each of length 50 years.

Results	Constant-saving-function, parameterization (3)	Unconditional Mean	Std. dev.	Mode	Median
e_R^b	EE error risky bond	1.00	5.23	1.05	0.76
e_X^c	EE error excess return	−0.87	5.21	−0.87	−0.73
RMSE	root mean square error	3.97	3.52	0.68	4.32
Observed random variables					
R_{t+1}^b	bill return (gross)	7.93	1.25	7.01	7.69
R_{t+1}^c	equity return (gross)	11.20	0.78	11.66	11.34
$R_{t+1}^c - R_{t+1}^b$	equity premium	3.27	1.44	3.27	3.33
$\ln(C_{t+1}/C_t)$	consumption growth	2.10	0.45	2.42	2.15
Parameter estimates					
$\hat{\beta}$	factor of time preference	0.94	0.50	1.00	0.99
$\hat{\gamma}$	coef. of relative risk aversion	267.66	520.80	5.00	3.56
$\widehat{e_R^b}$	EE error risky bond	−0.01	0.01	0.00	0.00
$\widehat{e_X^c}$	EE excess return	0.95	1.57	0.00	0.00
$\widehat{\text{RMSE}}$	root mean square error	0.67	1.11	0.00	0.00

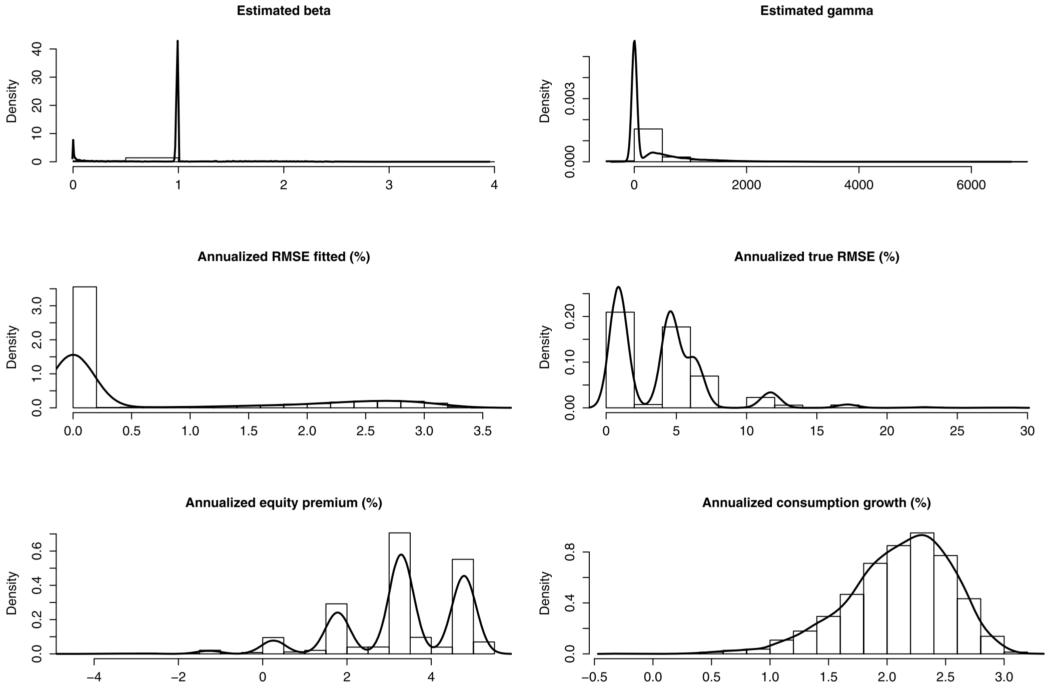


TABLE B.7. C-CAPM simulation results (long-run risk model). The table reports the simulated Euler equation (EE) errors and RMSE* (both annualized) for the standard C-CAPM observed at quarterly frequency in the endowment economy with long-run risk (cf. Appendix A.4) for a parameterization as in column (2) in Table B.3; the bond return, the equity return, the equity premium and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 7.5$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5000 Monte Carlo sample paths, each of length 50 years.

Results	Approximate solution parameterization (2)	Unconditional Mean	Std. dev.	Mode	Median
$R^b_{t+1} - E(R^b_{t+1})$	pricing error bond	0.00	0.50	−0.09	0.00
$R^d_{t+1} - E(R^d_{t+1})$	pricing error risky asset	0.00	0.84	0.16	−0.01
RMSE*	root mean square error	0.61	0.44	0.27	0.51
Observed random variables					
R^b_{t+1}	bill return	2.85	0.51	2.77	2.85
R^d_{t+1}	equity return	4.02	0.85	3.98	4.01
$R^d_{t+1} - R^b_{t+1}$	equity premium	1.17	0.47	1.08	1.17
$\ln(C_{t+1}/C_t)$	consumption growth	1.76	0.85	1.65	1.76
Parameter estimates					
$\hat{\beta}$	factor of time preference	1.05	0.05	0.99	1.04
$\hat{\gamma}$	coef. of relative risk aversion	16.03	6.67	14.05	15.78
$\widehat{e^b_R}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e^c_X}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{\text{RMSE}}$	root mean square error	0.00	0.00	0.00	0.00

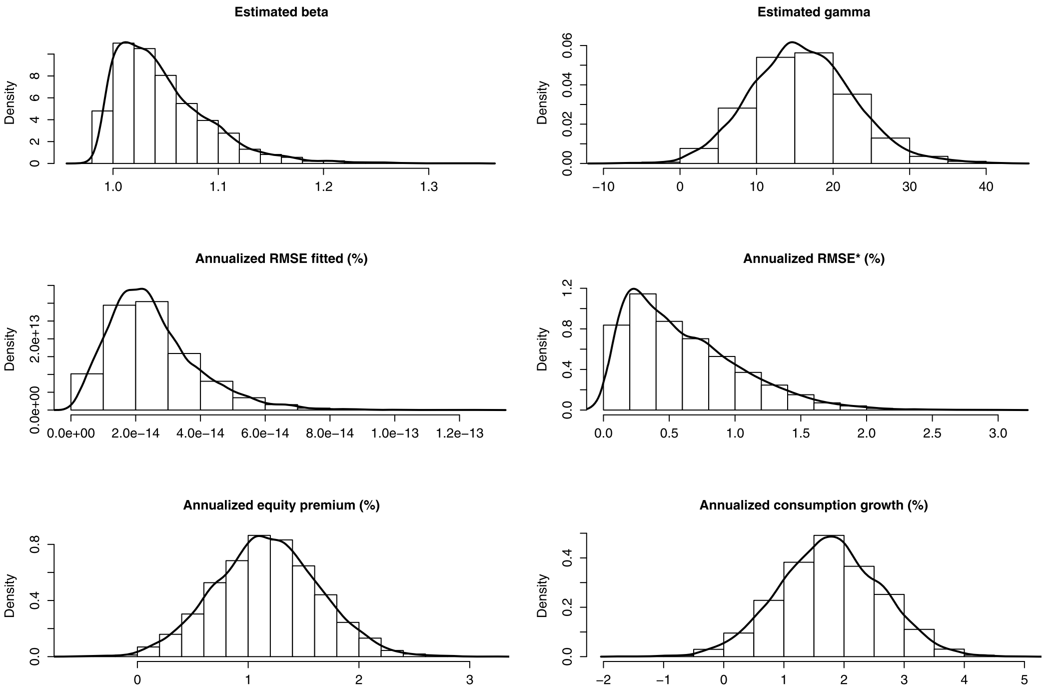
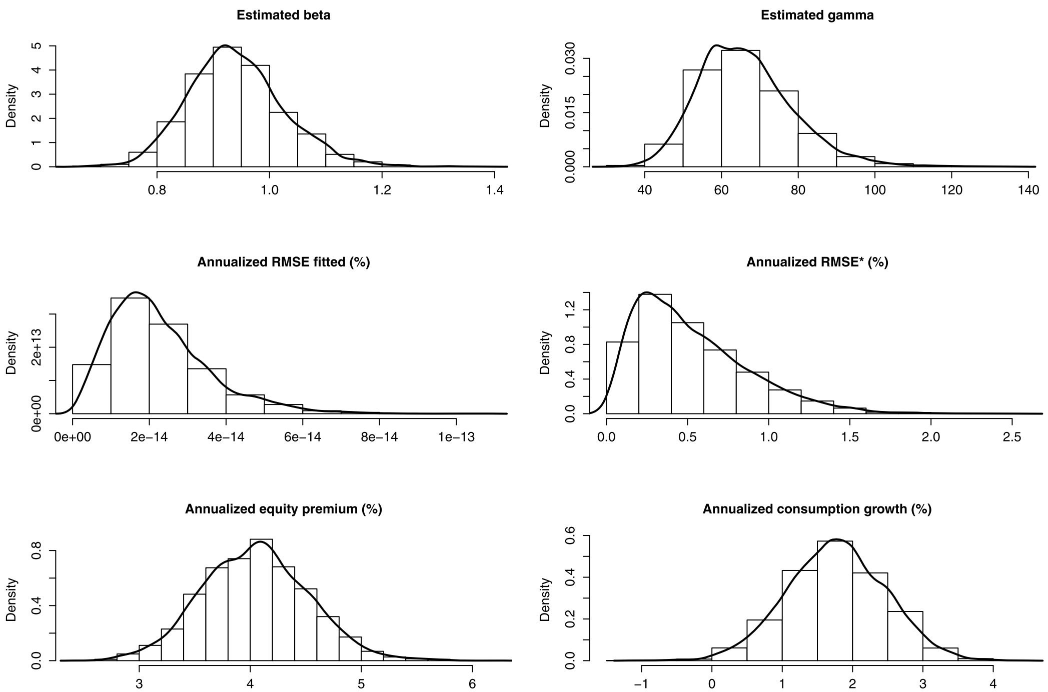


TABLE B.8. C-CAPM simulation results (long-run risk model). The table reports the simulated Euler equation (EE) errors and RMSE* (both annualized) for the standard C-CAPM observed at quarterly frequency in the endowment economy with long-run risk (cf. Appendix A.4) for a parameterization as in column (4) in Table B.3; the bond return, the equity return, the equity premium, and consumption growth (all annualized); and the GMM estimates of $\phi = (\beta, \gamma)^\top$ with $\beta = 0.98$ and $\gamma = 30$ based on moments (15), and the estimated EE errors and RMSE (both annualized). Simulated data is generated using 5000 Monte Carlo sample paths, each of length 50 years.

Results	Approximate solution parameterization (4)	Unconditional Mean	Std. dev.	Mode	Median
$R^b_{t+1} - E(R^b_{t+1})$	pricing error bond	0.00	0.46	0.22	0.01
$R^d_{t+1} - E(R^d_{t+1})$	pricing error risky asset	0.00	0.70	−0.14	−0.01
RMSE*	root mean square error	0.53	0.35	0.24	0.45
Observed random variables					
R^b_{t+1}	bill return	0.57	0.46	0.80	0.58
R^d_{t+1}	equity return	4.62	0.70	4.49	4.61
$R^d_{t+1} - R^b_{t+1}$	equity premium	4.05	0.48	4.07	4.04
$\ln(C_{t+1}/C_t)$	consumption growth	1.77	0.70	1.71	1.76
Parameter estimates					
$\hat{\beta}$	factor of time preference	0.94	0.09	0.92	0.94
$\hat{\gamma}$	coef. of relative risk aversion	66.42	12.49	65.50	65.09
$\widehat{e^b_R}$	EE error risky bond	0.00	0.00	0.00	0.00
$\widehat{e^c_X}$	EE excess return	0.00	0.00	0.00	0.00
$\widehat{\text{RMSE}}$	root mean square error	0.00	0.00	0.00	0.00



References

Breeden, D. T. (1986), "Consumption, production, inflation and interest rates—a synthesis." *Journal of Financial Economics*, 16, 3–39. [2]

Co-editor Tao Zha handled this manuscript.

Manuscript received 29 October, 2019; final version accepted 23 June, 2021; available online 3 August, 2021.